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# A representative consumer theorem for discrete choice models in networked markets

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ABSTRACT

We provide an alternative way to model sequential decision processes, which is consistent with the random utility maximization hypothesis and the existence of a representative agent. Our result is stated on terms of a direct utility representation, and it does not depend on parametric assumptions.



## 1. Introduction

In this paper, we propose a way of modeling sequential discrete decision processes, which is consistent with the random utility hypothesis and the existence of a representative agent. In particular, our approach is based on a network representation for the consumers' decision process and dynamic programming.[1] Combining the aforementioned elements, we show that a demand system for hierarchical or sequential decision processes can be obtained as the outcome of the utility maximization by a representative agent. Our result differs from previous findings in two important aspects. First, our result is in terms of a *direct* utility representation, whilst most of the results available in discrete choice theory are based on an *indirect* utility approach.[2] Second, and most important, our result does not depend on parametric assumptions on the random components associated to the utilities of different choices. We only require the mild condition that the distribution of the unobserved components must be absolutely continuous with respect to the Lebesgue measure. Thus given its generality, our approach and result can be useful to study a demand system with complex substitution patterns among the utilities associated with different choices.

An important feature of our result is that when we assume the specific double exponential distribution for the unobserved components, we show that the nested logit model can be seen as a particular case of our approach. In particular, we show that a sequential logit under specific parametric constraints coincides with the nested logit model (McFadden, 1978a,b, 1981).[3] This result generalizes previous findings in Borsch-Supan (1990), Konning and Ridder (1993), Herriges and Kling (1996), Verboven (1996), Konning and Ridder (2003), and Gil-Molto and Hole (2004). All of these papers impose parametric constraints in order to be consistent with the random utility maximization. Our result shows that such constraints can be avoided using the assumption of sequential decision making.

Finally, from an applied perspective we point out that our results can be useful to carry out welfare analysis in networked markets, where the standard discrete choice theory may not apply.

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1 The idea of analyzing discrete choice models using a network representation is also considered in Daly and Bierlaire (2006). They derive parametric constraints such that the class of generalized extreme value models can be implemented in decision processes represented through a network. Their results can be considered as special cases of our Theorem 1 in Section 2.

2 For a survey of these results see Anderson et al. (1992).

3 It is worth pointing out that the nested logit model does not need to be interpreted as a sequential decision process. In fact, its standard justification is based on preference correlation structure (see, e.g., Anderson et al., 1992).

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For example, our results can be applied to bundling decisions, merger analysis, or compatibility among goods in networked markets.[4]

The paper is organized as follows. Section 2 presents the model. Section 3 presents the main result of this paper, and Section 3.1 discusses the logit case. The Appendix A contains the proofs.

## 2. The model

Let $G = (N, A)$ be a *directed* graph with $N$ being the set of nodes and $A$ the set of links respectively.[5] Without loss of generality, we assume that the graph $G$ has a single origin–destination pair, where $o$ and $t$ stand for the origin node and destination node respectively.

We identify the set $N$ as the set of decision nodes faced by consumers, and the set $A$ is identified as the set of the available goods in this economy, i.e., the good $a$ is represented by the link $a \in A$.[6] Thus, starting in the origin $o$, consumers can choose bundles of goods through the choice of links on $A$. The destination $t$ is interpreted as the node that is reached once consumers have chosen their desired bundles of goods, and then they leave the market.

For each good $a \in A$, consumers' valuation is represented by $\theta_a \in \mathbb{R}_{++}$. Similarly, $p_a \in \mathbb{R}_+$ is the price associated to good $a$. Thus, the utility for good $a$ may be written as $u_a = \theta_a - p_a$. We assume that there exists a continuum of users with unitary mass. According to this, let $d = (d_a)_{a\in A}$ a non-negative flow vector, where $d_a \geq 0$ denotes the demand for good $a$. Any flow vector $d$ must satisfy the flow conservation constraints

$$\sum_{a\in A_i^-} d_a = \sum_{a\in A_i^+} d_a \quad \forall i \in N, \tag{1}$$

where $A_i^-$ denotes the set of links ending at node $i$, and $A_i^+$ denotes the set of links starting at node $i$. The set of feasible flows is denoted by $\mathcal{D}$.

It is worth emphasizing that in this paper we interpret each path in the graph as a bundle of goods.[7] This interpretation allows us to see the goods within a bundle as complements, and different paths can be viewed as substitute goods.

In order to introduce heterogeneity into the model, we assume that consumers are *randomly* drawn from a large population. According to this, the *random utility* $\widetilde{u}_a$ may be defined as

$$\widetilde{u}_a = u_a + \epsilon_a \quad \forall a \in A_i^+,\ i \in N,$$

with $\{\epsilon_a\}_{a\in A}$ being a collection of absolutely continuous random variables with $\mathbb{E}(\epsilon_a) = 0$ for all $a$. The random variables $\epsilon_a$'s take into account the heterogeneity within the population. In particular, these random variables represent the variability of the valuation $\theta_a$'s.

In this networked market, consumers choose the optimal bundle of goods in a recursive way. In particular, at each node consumers choose a good considering its utility plus the continuation value associated to their choices. Formally, at each node $i \neq d$ we define the random utility $\tilde{V}_a$ as

$$\tilde{V}_a = V_a + \epsilon_a \tag{2}$$

with $V_a = u_a + \varphi_{j_a}(V)$ and $\varphi_{j_a}(V) \equiv \mathbb{E}(\max_{b\in A_{j_a}^+}\{V_b + \epsilon_b\})$, where $j_a$ denotes that node $j_a$ has been reached using the link $a$.

Regarding Eq. (2) two remarks are important. First, Eq. (2) makes explicit the recursive nature of the consumers' choice process. In particular, consumers reaching node $i$ observe the realization of the random variables $\tilde{V}_a$, and choose the link $a \in A_i^+$ with the highest utility, taking into account the current utility $u_a$ plus the *continuation value* $\varphi_{j_a}(V)$.[8]

The second observation is that (2) makes explicit the assumption that a consumer makes sequential choices. In other words, consumers maximize utility solving a dynamic programming problem.[9]

From previous discussion, it follows that the expected flow $x_i$ entering node $i$ splits among the goods $a \in A_i^+$ according to

$$d_a = x_i \mathbb{P}(\underbrace{V_a + \epsilon_a}_{\widetilde{V}_a} \geq \underbrace{V_b + \epsilon_b}_{\widetilde{V}_b},\ \forall b \in A_i^+). \tag{3}$$

This *recursive discrete choice model* generates the following stochastic conservation flow equations

$$x_i = \sum_{a\in A_i^-} d_a. \tag{4}$$

Using a well known result in discrete choice theory,[10] Eqs. (3)–(4) may be expressed in terms of the gradient of the function $\varphi_i(\cdot)$. In particular, the conservation flow Eqs. (3) and (4) may be rewritten as

$$\begin{cases} d_a = x_i \dfrac{\partial \varphi_i(V)}{\partial V_a} \quad \forall a \in A_i^+, \\ x_i = \displaystyle\sum_{a\in A_i^-} d_a \end{cases} \tag{5}$$

where $\frac{\partial\varphi_i(V)}{\partial V_a} = \mathbb{P}(V_a + \epsilon_a \geq V_b + \epsilon_b,\ \forall b \in A_i^+)$.

Following the previous description, it is easy to see that consumers' choice process can be expressed as a Markov chain. In particular, once a consumer reaches a specific node, say node $i$, then consumers must make a choice among the goods available in the set $A_i^+$.

The following definition formalizes the notion of Markovian assignment in a networked market.[11]

**Definition 1.** Let $p \geq 0$ be a given price vector. A vector $d \in \mathbb{R}_+^{|A|}$ is a Markovian assignment if and only if the $d_a$'s satisfy the flow distribution Eq. (5) with $V$ solving $V_a = u_a + \varphi_{j_a}(V)$ for all $a \in A$.

We stress that the previous setting defines consumers utility in an indirect way. Assuming a specific distribution for the $\epsilon_a$'s, we can solve $V_a = u_a + \varphi_{j_a}(V)$ and find the demand vector. The next section establishes the main result of this paper: The Markovian assignment is equivalent to the demand system generated as the solution of a *direct utility function* by a representative consumer.

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4 For a survey of networked markets in Economics see Economides (1996).

5 In this paper we use dynamic programming techniques, so we do not need to assume that $G$ is acyclic. See Dasgupta et al. (2006, Chapter 6) for details.

6 The set $A$ can also be called as the set of choices.

7 We point out that the standard discrete model can be viewed as a particular case of our approach. In fact, we can define a network with the set of nodes $N$ consisting of just two nodes, where one node is the source and the other one is the sink, and a collection of $|A|$ parallel links representing the goods available in the market.

8 In the discrete choice literature the functions $\varphi_i(\cdot)$'s are known as the *inclusive values* at node $i \neq d$ (see McFadden, 1978a,b, 1981, and Anderson et al., 1992).

9 We point out that the idea of modeling discrete choices through a sequential process was first proposed by Ben-Akiva and Lerman (1985) in order to justify the nested logit model. Another paper exploiting the idea of sequential discrete choice models to analyze price competition among multi-product firms is the paper by Anderson and de Palma (2006).

10 For details see Chapter 2 in Anderson et al. (1992).

11 We point out that an equilibrium notion called Markovian traffic equilibrium has been introduced in Baillon and Cominetti (2008) and extended to oligopoly pricing problems in Melo (2011). However, neither Baillon and Cominetti (2008) nor Melo (2011) analyze the problem that is studied in this paper.

## 3. Main result

For the networked market described in Section 2, we consider there exists a representative consumer endowed with income $Y \in \mathbb{R}_{++}$. There is a numeraire good which is indexed by 0, and its price $p_0$ is normalized to unity. The budget constraint for the representative consumer is given by

$$B(p, Y) = \left\{ (d, d_0) \in \mathbb{R}_{+}^{|A|+1} : \sum_{a \in A} p_a d_a + d_0 \leq Y \right\} \tag{6}$$

where $d = (d_a)_{a \in A}$ is the demand vector for the good at the network, and $d_0$ is the demand for good 0. We recall that the demand $d$ must satisfy the flow constraint (1).

**Theorem 1.** *A representative consumer's utility function consistent with the Markovian assignment is given by*

$$U(d) = \begin{cases} \sum_{a \in A} \theta_a d_a + d_0 - \sum_{i \in N} \chi_i(d), & \text{s.t. } (1); \\ -\infty, & \text{otherwise} \end{cases} \tag{7}$$

*where* $\chi_i(d) = \sup_V \left\{ \sum_{a \in A_i^+} (V_a - \varphi_i(V)) d_a \right\}$.

It is worth emphasizing four important features of Theorem 1.

First, we note that Theorem 1 does not require independence of the random variables $\epsilon_a$'s. Thus this result can deal with complex correlation patterns among different alternatives.

Second, Theorem 1 is based on a direct utility function for the representative consumer. In particular, the direct utility function in expression (7) encapsulates two different components. The first component, given by the linear term $\sum_{a \in A} \theta_a d_a + d_0$, expresses the utility derived from the consumption of $(d, d_0)$ in the absence of interaction among goods. Furthermore, the valuation parameters $\theta_a$'s can be viewed as measuring the intrinsic contribution of good $a$ to the total utility. The second effect is given by $-\sum_{i \in N} \chi_i(d)$, which expresses the *variety-seeking behavior* of the representative consumer[12]. The interpretation of variety-seeking behavior has been given in Anderson et al. (1988).

Third, we note that for the simple case where the number of goods is $|A|$, and just two nodes, an origin and a destination, Theorem 1 provides a direct representation without imposing parametric assumptions on the collection of $\epsilon_a$'s. For this simple case, Theorem 1 generalizes previous results in discrete choice theory.[13]

Finally, we note that Theorem 1 can be extended to the case of endogenous consumption. In particular, instead of considering the linear component $\sum_{a \in A} \theta_a d_a$, we can consider a strictly concave function $F(d; \theta)$ with continuous second derivatives. Using this function $F(d; \theta)$, the strict concavity of the optimization problem holds, so that we can apply the same reasoning given in the proof of Theorem 1.

### 3.1. The sequential logit case

In theoretical and applied work, the nested logit model is the leading case for modeling markets with a tree or network structure. In this section we show that the nested logit is a particular case of Theorem 1, which is obtained assuming that at each node $i$ the $\epsilon_a$'s are *i.i.d.* random variables follow a double exponential distribution.

**Proposition 1.** *Assume that at each node i the random variables $\epsilon_a$'s are i.i.d. following a double exponential distribution with location parameter $\beta_i \in \mathbb{R}_{++}$. Then, a representative consumer's utility function consistent with the Markovian assignment is given by*

$$U(d) = \begin{cases} \sum_{a \in A} \theta_a d_a + d_0 - \sum_{i \in N} \chi_i(d), & \text{s.t. } (1); \\ -\infty, & \text{otherwise} \end{cases} \tag{8}$$

*where* $\chi_i(d) = \frac{1}{\beta_i} \left( \sum_{a \in A_i^+} d_a \log d_a - \sum_{a \in A_i^+} d_a \log \left( \sum_{a \in A_i^+} d_a \right) \right)$.

We stress that Proposition 1 generates a demand system based on a sequential logit model. However, after some simple algebra, it is possible to show that the choice probabilities in Proposition 1 can be written as a nested logit. In particular, we can find the explicit parametric constraints on the $\beta_i$'s such that Proposition 1 yields a demand system based on a nested logit model.[14]

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## Appendix. Proofs

**Proof of Theorem 1.** Noting that the utility function $U(d)$ is strictly concave, the first order conditions are necessary and sufficient for finding a maximum. Using this fact, we can write the Lagrangian for the consumer's optimization problem as

$$\mathcal{L} = \sum_{a \in A} \theta_a d_a + d_0 - \sum_{i \in N} \chi_i(d) + \lambda \left[ Y - \sum_{a \in A} p_a d_a - d_0 \right] + \sum_{i \in N} \mu_i \left[ \sum_{a \in A_i^-} d_a - \sum_{a \in A_i^+} d_a \right] + \sum_{a \in A} \lambda_a d_a.$$

The multipliers $\mu_i$'s and $\lambda \in \mathbb{R}$ correspond to constraints (1) and (6), and $d_a \geq 0$ respectively. For a stationary point we get $\lambda = 1$, $u_a^* = \theta_a - p_a$ and $\zeta \in \partial(-\chi(v))$ with $\zeta_a = u_a^* + \mu_{j_a} - \mu_{i_a}$ and $\chi(d) = \sum_{i \in N} \chi_i(d)$. For the multipliers $\lambda_a$'s we simply set $\lambda_a = 0$ for all $a \in A$. Taking $\mu_i = \varphi_i((u_a^* + \varphi_{j_a}(V))_{a \in A})$, and combining (1) and (5) we get

$$d_a = \frac{\partial \varphi_{i_a}}{\partial d_a}(V) \sum_{a \in A_i^+} d_a,$$

which shows that $V$ is an optimal solution for $-\chi(d)$. Therefore, setting $g_a = \varphi_{i_a}(V) - V_a$ we get $g \in \partial(-\chi(d))$. Combining $\varphi_{i_a}(V)$ with $V_a = u_a^* + \varphi_{j_a}(V^*)$, it follows that $\zeta = g \in \partial(-\chi(d))$ as required. □

**Proof of Proposition 1.** We can write the Lagrangian for the consumer's optimization problem as

$$\mathcal{L} = \sum_{a \in A} \theta_a d_a + d_0 - \sum_{i \in N} \frac{1}{\beta_i} \left( \sum_{a \in A_i^+} d_a \log d_a \right.$$

12 The terms $-\sum_{i \in N} \chi_i(d)$ can be viewed as a generalized entropy. In Section 3.1 this interpretation is clearer when we assume that the $\epsilon_a$'s follow a double exponential.

13 For a survey of the results of representative agents and demand system in discrete choice models see Anderson et al. (1992, Chapter 3).

14 The details proving this fact are available upon request.

$$- \sum_{a\in A_i^+} d_a \log\left(\sum_{a\in A_i^+} d_a\right)\Bigg) + \lambda\left[Y - \sum_{a\in A} p_a d_a - d_0\right]$$
$$+ \sum_{i\in N}\mu_i\left[\sum_{a\in A_i^-} d_a - \sum_{a\in A_i^+} d_a\right] + \sum_{a\in A}\lambda_a d_a.$$

As in Theorem 1, the multipliers $\mu_i$'s and $\lambda \in \mathbb{R}$ correspond to constraints (1) and (6), and $d_a \geq 0$ respectively. Taking the first order condition and setting $\lambda_a = 0$, we get

$$\frac{\partial \mathcal{L}}{\partial d_0} = 1 - \lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial d_a} = \theta_a - \lambda p_a - \frac{1}{\beta_i}\left[\log d_a - \log\left(\sum_{a\in A_i^+} d_a\right)\right] + \mu_{j_a} - \mu_{i_a} = 0 \quad \forall i \in N.$$

Combining (1) and (5), and after some simple algebra we find that

$$d_a = x_i \frac{e^{\beta_i(\theta_a - p_a + \varphi_{j_a})}}{\sum_{b\in A_i^+} e^{\beta_i(\theta_b - p_b + \varphi_{j_b})}} \quad \forall a \in A_i^+,$$

which is equivalent to

$$d_a = x_i \frac{e^{\beta_i V_a}}{\sum_{b\in A_i^+} e^{\beta_i V_b}} \quad \forall a \in A_i^+,$$

and the conclusion follows at once. □

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