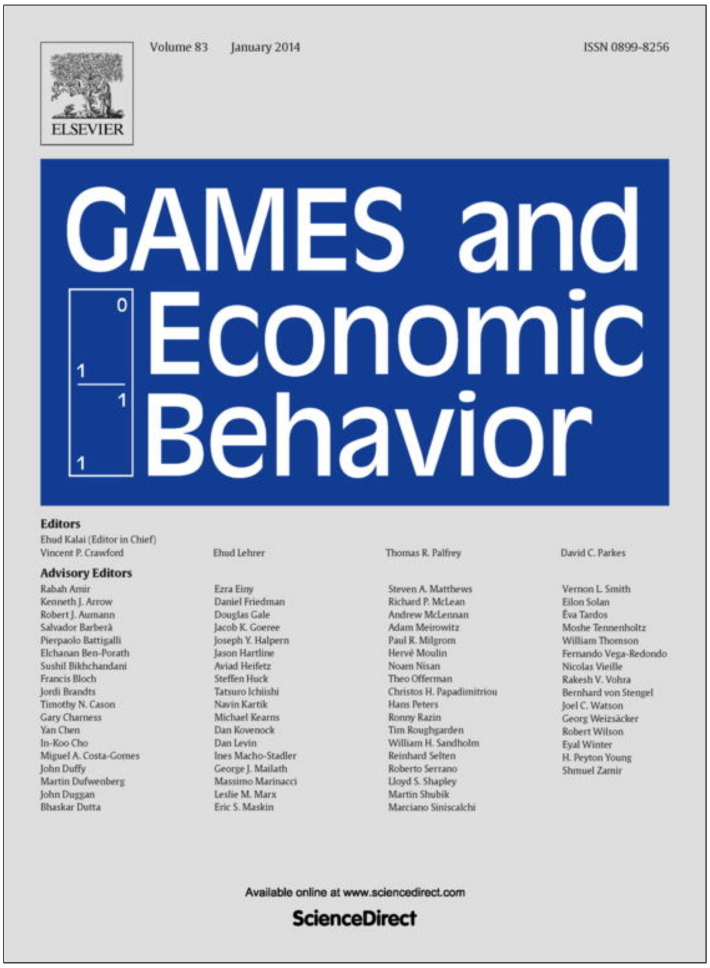

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Price competition, free entry, and welfare in congested markets

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ABSTRACT

In this paper we study the problem of price competition and free entry in congested markets. In particular, we consider a network with multiple origins and a common destination node, where each link is owned by a firm that sets prices in order to maximize profits, whereas users want to minimize the total cost they face, which is given by the congestion cost plus the prices set by firms. In this environment, we introduce the notion of *Markovian Traffic Equilibrium* to establish the existence and uniqueness of a pure strategy price equilibrium, without assuming that the demand functions are concave nor imposing particular functional forms for the latency functions. We derive explicit conditions to guarantee existence and uniqueness of equilibria. Given this existence and uniqueness result, we apply our framework to study entry decisions and welfare, and establish that in congested markets with free entry, the number of firms exceeds the social optimum.

1. Introduction

In many environments, such as communication networks in which network flows are allocated, or transportation networks in which traffic is directed through the underlying road architecture, congestion plays an important role in terms of efficiency. In fact, over the last decade the phenomenon of congestion in traffic networks has received attention in a number of different disciplines: economics, computer science, and operations research.

The main question is how to achieve a socially optimal outcome, which is intimately linked to the assessment of congestion effects. This feeds into the identification of socially optimal regulatory actions in such markets. Indeed, a social planner may use a sort of economic mechanisms in order to induce users' behavior toward the socially optimal outcome. In fact, since the seminal work of Pigou (1920), it is well known that an efficient outcome in a network subject to congestion can be reached through the *centralized implementation* of a toll scheme based on the principle of *marginal cost pricing*. Under this mechanism users pay for the negative externality that they impose on everybody else.

Concretely, under a Pigouvian tax scheme users face two sources of cost: one due to the congestion cost and the second due to the toll. Nonetheless, Pigou's solution is hard to implement in practice, because it requires that the social planner charges the tolls in a centralized way, which from a practical and computational perspective is a very complex task. Thus, the natural alternative is to consider a market-based solution, where every route (or link) of the network is owned by independent firms who compete setting prices in order to maximize profits.[1]

[1] For an early discussion of price mechanisms in congested networks, we refer the reader to Luski (1976), Levhari and Luski (1978), Reitman (1991), and MacKie-Mason and Varian (1995).

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Despite the relevance of and the increasing interest in implementing decentralized pricing mechanisms to reduce and control congestion in networks, little is known about the theoretical properties of such solutions for general class of network topologies. Indeed, little is known about the existence and uniqueness of equilibrium prices for general classes of network topologies.

In addition to the problem of the existence and uniqueness of a price equilibrium, a second problem that is raised in congested markets is the analysis of free entry and welfare. In particular, every firm can be viewed as a link, so the number of firms that enter the market will determine the network topology. Thus the socially optimal topology can be identified with the optimal number of firms in the market. Similarly to the study of existence and uniqueness of a price equilibrium, little is known about the free entry problem in general networks.

1.1. Our contribution

In this paper we develop and study a general oligopoly model in a network subject to congestion effects. Our contribution is threefold. First, we study oligopoly pricing in congested networks exploiting an alternative notion of equilibrium in traffic networks, which we denote as *Markovian Traffic Equilibrium*. Our equilibrium concept is based on the idea that users choose their optimal paths in a recursive way. The idea that users can find their optimal paths in a recursive way turns out to be different to the standard notion of Wardrop equilibria. In particular, our equilibrium concept allows for heterogeneous users and general network topologies.[2] We are not aware of previous papers studying price competition in congested markets using the notion of Markovian Traffic Equilibrium.

Second, we show the *existence* and *uniqueness* of a pure strategy price equilibrium. Our result is general: we do not assume that demand functions are concave nor impose particular functional forms for the latency functions (congestion costs) as is commonly assumed in the extant literature. We derive *explicit conditions* to guarantee existence and uniqueness of equilibria. We stress that our existence and uniqueness result does not rely on a specific network topology. In fact, our result applies to any directed acyclic network with multiple origins and a common destination node.

Our third contribution is the study of entry decisions and welfare in congested markets. We show that the number of firms that enter the network exceeds the social optimum. In terms of network design the excess entry result means that the observed topology will not be the socially optimal. Because we obtain this result for a general network, we think of that our excess entry result may be useful in studying problems of optimal design of networks.

Formally, we study a network with multiple origins and a common destination node, where every link is owned by a firm that sets prices in order to maximize profits. In this environment, users face two sources of cost: the congestion cost plus the price set by the firms. The congestion in every link is captured by a latency function, which is strictly increasing in the number of users utilizing it. In order to solve the users' problem, we adapt the Markovian Traffic model proposed by Baillon and Cominetti (2008), to the study of price competition in congested networks. This Markovian model is based on *random utility models* and *dynamic programming*. The use of random utility models allows for heterogeneity in users' behavior, i.e., instead of assuming homogeneous users, we model the utility of choosing a certain route as a random variable. In addition, and considering the stochastic structure of users' utilities, we assume that users solve a dynamic programming problem in order to construct the optimal path in a recursive way. Thus, at each node users consider the utility derived from the available links plus the continuation values associated to each link.

Furthermore, the introduction of random utility models has the advantage of generating a *demand system*, which shows how prices and congestion externalities induce users' choices.[3]

Combining the previous elements, we solve a complete information two stage game, which can be described as follows: In the first stage, firms owning the links maximize profits setting competitive prices *a là* Bertrand. In the second stage, given firms' prices, users choose routes in order to maximize their utility, namely the cheapest route. We solve this game using backward induction, looking for a pure strategy sub-game perfect Nash equilibrium, which we call the *Oligopoly Price Equilibrium*.

Despite being a stylized model, an important real world situation where our framework may be useful is the case of road pricing in transportation networks. In particular, in transportation networks commuters from different locations (sources) want to travel to a common destination, let us say d. In order to reach the destination d, commuters must choose among different paths, where the total cost faced by commuters is given by the price charged by firms (operators) plus congestion. In real world transportation networks, the links of different paths are managed by different private operators, by local governments, or by a mix of private and public operators.[4] For example, the transportation network in Europe has received attention because commuters must choose paths across different countries, where the prices are set by a mix of different

[2] The assumptions of homogeneous users and very specific network topologies, like parallel serial link networks, have been recognized as one of the main limitations of the previous studies of pricing in congested networks. For a survey of the different models of pricing in congested networks we refer the reader to Nisan et al. (2007, Ch. 22).

[3] We stress that our approach differs with the one known as aggregation in oligopoly markets proposed by Caplin and Nalebuff (1991). The main technical difference is due to our demand system being defined in terms of a fixed point equation, which reflects the existence of congestion externalities in users' choices, while the results in Caplin and Nalebuff (1991) do not apply to the case of demand systems with externalities (positive or negative).

[4] For a detailed discussion of road pricing under different ownership schemes we refer the reader to Yang and Huang (2005) and Small and Verhoef (2007, Ch. 6).

public (government) and private operators.[5] In this context, information about the conditions of existence and uniqueness of a price equilibrium, and how the transportation network behaves under free entry provides useful information to regulators who want to know the effect on welfare and congestion levels of different pricing policies, different ownership regimes, investment in highways, and the effect of adding (or deleting) links on the network.[6]

1.2. Related work

The pricing game that we study in this paper is not new at all. In fact, this class of games is studied in Cachon and Harker (2002), Engel et al. (2004), Hayrapetyan et al. (2005), Acemoglu and Ozdaglar (2007), Baake and Mitusch (2007), Allon and Federgruen (2007, 2008), Chawla and Roughgarden (2008), and Weintraub et al. (2010). In order to establish the existence of an oligopoly price equilibrium, these papers assume the following: First, in order to describe users' behavior these papers make use of the concept of *Wardrop equilibria*, which establishes that utilities (costs) on all the routes actually used are equal, and greater (less) than those which would be experienced by a single user on any unused route. Second, these papers impose assumptions on the demand generated by users' behavior or assumptions on the class of latency functions. In particular, Cachon and Harker (2002), Engel et al. (2004), Hayrapetyan et al. (2005), and Weintraub et al. (2010) assume that the demand functions are concave (or log-concave) functions of the price charged by firms. Thanks to the concavity assumption, the previous papers show the existence of an oligopoly price equilibrium. On the other hand, the papers of Acemoglu and Ozdaglar (2007), Baake and Mitusch (2007), and Chawla and Roughgarden (2008) show the existence of a pure strategy equilibrium assuming that the latency functions are affine.

Moreover, all of the papers mentioned above, consider a simple network consisting of a single origin–destination pair, with a collection of parallel links. This specific network topology rules out some interesting examples from an applied point of view,[7] thus limiting the application of the available existence results.

The recent papers of Allon and Federgruen (2007, 2008), use random utility models to study price in competition in the context of queuing games, where the latency functions represent the waiting time that users must wait to be served. These papers establish the existence and uniqueness of an oligopoly equilibrium. However, the results in Allon and Federgruen (2007, 2008) differ from ours in two important aspects. First, Allon and Federgruen (2007, 2008) consider a network consisting of a single origin–destination pair with parallel links, thus ruling out important cases from an applied perspective. Second, Allon and Federgruen (2007, 2008) do not study the entry and welfare problem.

Regarding our result on free entry and welfare, similar findings in the context of traditional oligopoly theory can be found in Mankiw and Whinston (1986) and Anderson et al. (1995). These papers do not, however, deal with network structures on congestion; features that are crucial components of our result. For the case of congested networks, a similar result to ours can be found in the recent paper of Weintraub et al. (2010) for the particular case of a network with a single pair source–sink and assuming parallel links. Summarizing, our results can be viewed as a generalization of previous findings of the free entry and welfare problem.

The rest of the paper is organized as follows: Section 2 presents the model. Section 3 studies the free entry and welfare problem. Section 4 shows the result of existence and uniqueness of a oligopoly price equilibrium for a general class of latency functions. Finally, Section 5 concludes. Proofs and technical lemmas are presented in Appendix A.

2. The model

Let $G = (N, A)$ be a *directed acyclic* graph representing a traffic network, with N being the set of nodes and A the set of *links* respectively. Let $d \in N$ be the destination node (or sink). For each node $i \neq d$, $g_i \geqslant 0$ denotes the numbers of users starting at that node. We interpret g_i as a continuum of users. For all $i \neq d$, we denote $\mathcal{R}_i$ as the set of available routes connecting node i with the destination node d. Every link a is represented by a convex and strictly increasing continuous latency function $l_a : \mathbb{R} \mapsto (0, \infty)$, which we assume to be twice continuously differentiable.

A flow vector is a nonnegative vector $v = (v_a)_{a \in A}$, where $v_a \geqslant 0$ denotes the mass of users using link a. Any flow vector v must satisfy the flow conservation constraint:

$$g_i + \sum_{a \in A_i^-} v_a = \sum_{a \in A_i^+} v_a, \quad \forall i \neq d, \tag{1}$$

where A_i^- denotes the set of links ending at node i, and A_i^+ denotes the set of links starting at node i. The set of feasible flows is denoted by V.

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[5] In fact, De Borger et al. (2005, 2007) analyze the case of tax competition motivated by the European transportation network.

[6] We note that in order to incorporate a public operator in our model, we need to modify the profit function that the public operator wants to maximize. For a detailed discussion of profit functions for public operators we refer the reader to Small and Verhoef (2007, Ch. 6).

[7] For instance, this specific topology rules out the case of hub-spoke networks.

We introduce firms into the network through the assumption that each link a is operated by a different firm that sets prices in order to maximize profits. In particular, firm a's profits are given by:

$$\pi_a(p, v_a) = p_a v_a, \quad \forall a \in A. \tag{2}$$

Profit maximization generates a nonnegative price vector p, $p = (p_a)_{a\in A}$.

In addition, and without loss of generality, we set the parameter $R > 0$ to be the users' reservation utility at each link a. Thus, given a flow v and a price vector p, the users' utility is given by:

$$u_a = R - p_a - l_a(v_a), \quad \forall a \in A.$$

In this environment, firms and users strategically interact in the following way: at every node $i \neq d$ the firms owning the set of links starting in node i set prices in order to maximize profits. Then, considering the price vector generated by firms' behavior, users choose routes in order to maximize their utility. The solution concept for this game is a sub-game perfect Nash equilibrium, which we shall refer to the Oligopoly Price Equilibria.

We look for an Oligopoly Price equilibrium using backward induction, i.e., given a price vector p, we solve the users' problem. Given the optimal solution for users, we solve the firms' problem.

It is worth noting that the previous framework is deterministic, so the notion of Wardrop equilibria turns out to be suited for solving the users' problem. Thus the firms' maximization profit considers the demand generated by this solution concept. This way of analysis has been traditional in the context of pricing in congested markets, and examples of its use are the papers of Acemoglu and Ozdaglar (2007), Engel et al. (2004), Hayrapetyan et al. (2005), Weintraub et al. (2010), Anselmi et al. (2011).

In this paper we propose an alternative model to study pricing in congested networks. In particular, we consider heterogeneous consumers, where one of the main features of our approach is that the users' optimal solution is based on the combination of random utility models and dynamic programming. The next section describes in detail these ideas.

2.1. Markovian traffic equilibrium

In this section we introduce our equilibrium concept for the users' problem, which is based on two important features. First, to solve the users' problem we introduce the idea of random utility, which takes into account the heterogeneity of users' preferences. Second, we use techniques borrowed from dynamic programming to find in a sequential way the optimal path for users. We now proceed to explain in detail our approach.

We introduce heterogeneity in the model assuming that users are *randomly* drawn from a large population having variable perceptions of the utility of each link a. According to this, the *random utility* $\widetilde{u}_a$ may be defined as

$$\widetilde{u}_a = u_a + \epsilon_a, \quad \forall a \in A,$$

with $\{\epsilon_a\}_{a\in A}$ being a collection of absolutely continuous random variables with $\mathbb{E}(\epsilon_a) = 0$ for all a. At least two justifications for introducing $\{\epsilon_a\}_{a\in A}$ can be given. The first explanation comes from the fact that at each link a, the random variable ϵ_a takes into account the variability of users' reservation utility. This means that at each link a we can model the reservation utility as a random variable defined as $R_a = R + \epsilon_a$, with $\mathbb{E}(R_a) = R$. A similar justification can be given if we model the congestion costs as random variables. Concretely, for any given flow vector v, at each link a we can consider the random cost defined as $\widetilde{l}_a(v_a) = l_a(v_a) + \epsilon_a$, where $\mathbb{E}(\widetilde{l}_a(v_a)) = l_a(v_a)$.[8] For all $i \neq d$, let $\mathcal{R}_i$ denote the set of routes connecting node i with destination d. Thus, for a route $r \in \mathcal{R}_i$, we define its utility as $\tilde{u}_r = \sum_{a\in r} \tilde{u}_a$, and therefore the optimal utility $\tilde{\tau}_i = \max_{r\in\mathcal{R}_i} \tilde{u}_r$ as well as the utility $\widetilde{z}_a = \widetilde{u}_a + \widetilde{\tau}_{j_a}$ can be rewritten as $\tilde{\tau}_i = \tau_i + \theta_i$ and $\tilde{z}_a = z_a + \epsilon_a$, where j_a denotes that node j has been reached using the link a, and $\mathbb{E}(\theta_i) = \mathbb{E}(\epsilon_a) = 0$. Each user traveling towards the final node, and reaching the node i, observes the realization of the variables $\tilde{z}_a$ and then chooses the link $a \in A_i^+$ with the highest utility. This process is repeated in each subsequent node giving rise a *recursive discrete choice model*, where the expected flow x_i entering node $i \neq d$ splits among the arcs $a \in A_i^+$ according to

$$v_a = x_i \mathbb{P}\big(\widetilde{z}_a \geqslant \widetilde{z}_b,\ \forall b \in A_i^+\big). \tag{3}$$

Furthermore, the recursive discrete choice model generates the following conservation flow equations

$$x_i = g_i + \sum_{a\in A_i^-} v_a. \tag{4}$$

[8] We can also consider the case where the utilities of every link are deterministic and the variability within the population is captured by the distribution of tastes in regard each link. Both justifications yield the same mathematical structure in terms of expected demand. For a detailed discussion see Anderson et al. (1992).

Using a well-known result in discrete choice theory (see Anderson et al., 1992), Eqs. (3)–(4) may be expressed in terms of the gradient of the function $\varphi_i(\cdot)$ defined as $\varphi_i(z) \equiv \mathbb{E}(\max_{a\in A_i^+}\{z_a+\epsilon_a\})$.[9] In particular, the conservation flow equations (3) and (4) may be rewritten as

$$\begin{cases} v_a = x_i \dfrac{\partial \varphi_i}{\partial z_a}(z), \quad \forall a \in A_i^+, \\ x_i = g_i + \displaystyle\sum_{a\in A_i^-} v_a; \end{cases} \tag{5}$$

where $\frac{\partial \varphi_i}{\partial z_a}(z) = \mathbb{P}(\widetilde{z}_a \geqslant \widetilde{z}_b, \forall b \in A_i^+)$.

Given the recursive structure of the problem, we may write the corresponding Bellman's equation in the form $\widetilde{\tau}_i = \max_{a\in A_i^+} \widetilde{z}_a$ using $\widetilde{z}_a = \widetilde{u}_a + \widetilde{\tau}_{j_a}$. Thus, taking expectation we get

$$z_a = u_a + \varphi_{j_a}(z) \tag{6}$$

or in terms of the variables τ_i

$$\tau_i = \varphi_i\big((u_a + \tau_{j_a})_{a\in A}\big). \tag{7}$$

In order to simplify our analysis, we assume the following two conditions for the random variables $\{\epsilon_a\}_{a\in A}$.

Assumption 1. For all $i \neq d$ and for all $r \in \mathcal{R}_i$, the collection of random variables $\{\epsilon_a\}_{a\in r}$ are independent.

Assumption 2. For each node $i \neq d$, the collection of random variables $\{\epsilon_a\}_{a\in A_i^+}$ are i.i.d. following a Gumbel (double exponential) distribution with localization parameter $0 < \beta_i < \infty$.[10]

We stress that Assumption 1 rules out the possibility that the ϵ_a in the same path may exhibit dependence. In terms of our model, Assumption 1 implies that realizations of ϵ_as do not affect $\varphi_{j_a}(\cdot)$. However, Assumption 1 does not impose independence among different paths.[11]

Regarding Assumption 2, we note that is made for expositional simplicity, but all our results hold for a general collection of random variables $\{\epsilon_a\}_{a\in A_i^+}$ with the technical requirement that the density of each ϵ_a is twice differentiable. In particular, we can allow for very complex patterns of correlation among the ϵ_as at each node $i \neq d$.[12] Assumption 2 allows us to write the functions $\varphi_i(z)$ in a closed form (see Anderson et al., 1992):

$$\varphi_i(z) = \frac{1}{\beta_i}\log\left(\sum_{b\in A_i^+} e^{\beta_i z_b}\right), \quad \forall i \neq d.$$

Using this log-sum formula, it follows that $\frac{\partial \varphi_i}{\partial z_a}(z) = \frac{e^{\beta_i z_a}}{\sum_{b\in A_i^+} e^{\beta_i z_b}}$, i.e., $\frac{\partial \varphi_i}{\partial z_a}(z)$ is the logit-choice rule.

From Eqs. (6) and (7), it follows that for every price vector p, users recursively find the routes with the highest utility through the solution of a dynamic programming problem. This means that instead of choosing routes, users recursively choose links considering the utilities and the continuation values at every node. The method of solving recursively the users' problem turns out to be completely different from the standard notion of Wardrop equilibrium.[13] We shall call this solution concept *Markovian Traffic Equilibrium*. Its formal definition is:

Definition 1. Let $p \geqslant 0$ be a given price vector. A vector $v \in \mathbb{R}_+^{|A|}$ is a Markovian Traffic Equilibrium (MTE) iff the v_a's satisfy the flow distribution equation (5), with z solving (6).

Definition 1 formalizes the idea that for a given price vector p, users solve the associated dynamic programming problem such that the flow vector v is distributed in an optimal way. That is, the flow v is distributed such that users' utility

---

9 The functions $\varphi_i(\cdot)$s are known as the social surplus in the literature of discrete choice models. In particular, this definition is introduced in McFadden (1981).

10 The Gumbel (or double exponential) distribution for ϵ_a is given by $\mathbb{P}(\epsilon_a \leqslant x) = \exp(-\exp(-\{\beta_i x + \gamma\}))$ where γ is Euler's constant and $0 < \beta_i < \infty$.

11 We note that Assumption 1 is weaker than the assumption of independent paths which is widely used in the transportation literature under the label of stochastic user equilibrium. We refer the reader to Yang and Huang (2005) for an application of stochastic user equilibrium and pricing in the context of transportation networks.

12 We note that this independence assumption does not rule out the possibility of correlated paths. In fact, under Assumption 2 all the paths sharing common links are correlated.

13 Recall that a Wardrop equilibrium establishes that utilities (costs) on all the routes actually used are equal, and greater (less) than those which would be experienced by a single user on any unused route.

is maximized. The notion of MTE has been introduced in Baillon and Cominetti (2008), and it generalizes the concept of *stochastic user equilibrium* considered by Daganzo and Sheffi (1977) and Fisk (1980).[14] In this paper we use the concept of MTE because it allows us to introduce heterogeneity within users through the stochastic terms ϵ_as. More importantly, the notion of MTE allows us to study price competition among firms exploiting the recursive structure in the users' decisions. In particular, at every node $i \neq d$, and thanks to the Markovian structure, we can study price competition just considering the firms owning the links at every node. In other words, we exploit the recursive structure of users' problem to decompose the problem of price competition for the whole network into a collection of local oligopoly pricing problems at each node $i \neq d$. In addition, the Markovian structure makes it possible to study general network topologies in a simple fashion. Example 1 below shows how MTE works for a small network.

Example 1. This example shows how the notion of MTE works. Consider the following network (Fig. 1):

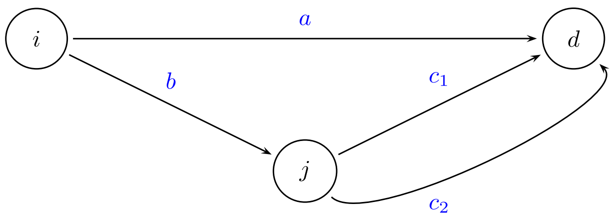

Fig. 1. Finding an MTE.

The set of nodes is $N = \{i, j, d\}$, where i and d represent the origin and destination node respectively. The set of links is represented by $A = \{a, b, c_1, c_2\}$. For each link k the users' cost is given by $p_k + l_k(v_k)$ with $k = a, b, c_1, c_2$, where v_k is the flow of users choosing link k. For a fixed price vector p, MTE requires that users solve a dynamic programming problem. According to this, at node i the users compare the cost of links a and b taking into account the associated continuation values. Thus, the users will choose link b if and only if $u_b + \tau_{j_b} + \epsilon_b \geqslant u_a + \epsilon_a$. Conditional on the choice of link b, the users reach node j. Then, they must choose between c_1 and c_2, considering the total costs and the associated continuation values. Finally, and noting that for this case the associated continuation values to c_1 and c_2 are zero, the users will choose link c_1 if and only if $u_{c_1} + \epsilon_{c_1} \geqslant u_{c_1} + \epsilon_{c_2}$. The same logic applies to how other paths are chosen. The key point is that the optimal path is constructed in a recursive fashion.

2.2. Existence and uniqueness of an MTE

Now we are ready to characterize the MTE as the *unique* solution of a concave optimization program.

Proposition 1. *Given any price vector $p \geqslant 0$, the MTE is the unique optimal solution v^* of*

$$\max_{v \in V} \left\{ \sum_{a \in A} (R - p_a) v_a - \sum_{a \in A} \int_0^{v_a} l_a(s)\, ds - \chi(v) \right\}, \qquad (P)$$

where $\chi(v) = \sum_{i \neq d} \frac{1}{\beta_i} [\sum_{a \in A_i^+} v_a \ln v_a - (\sum_{a \in A_i^+} v_a \ln(\sum_{a \in A_i^+} v_a))]$.

We stress three important points regarding Proposition 1. First, we point out that the result of Proposition 1 is a slight variation of Theorem 2 in Baillon and Cominetti (2008). We have adapted their result to incorporate the price vector p. Second, the result in Proposition 1 is the stochastic version of the classical characterization for Wardrop equilibria introduced by Beckmann et al. (1956) (see also Ch. 18 in Nisan et al., 2007). In fact, in the deterministic case with $\varphi_i(z) = \min\{z_a\colon a \in A_i^+\}$ we get $\chi(v) \equiv 0$, so that the characterization in Proposition 1 coincides with the one given by Beckmann et al. (1956).[15] Intuitively, the variational problem in Proposition 1 can be viewed as a perturbed version of the deterministic problem, where the perturbation is given by $\chi(v)$, which takes into account the heterogeneity of users' utilities.[16]

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[14] For a discussion of different equilibrium concepts used in traffic networks see the recent survey by Correa and Stier-Moses (2010).

[15] We point out that Beckmann et al. (1956)'s characterization of Wardrop equilibria provide uniqueness of an optimal flow over links, but their decomposition of the optimal flow over paths is not unique. The result in Proposition 1 establishes the uniqueness of an optimal flow over links and paths.

[16] We note that $\chi(v) = 0$ means that $\beta_i \to \infty$ for all i. This follows because under Assumption 2, for all $i \neq d$, $a \in A_i^+$ we get $\mathbb{V}(\epsilon_a) = \frac{\pi}{\beta_i \sqrt{6}}$. Thus the homogeneous case corresponds to a situation where the random variables ϵ_a's are degenerate with mean and variance equal to zero. In particular, as long as $\beta_i \to \infty$ for all $i \neq d$, the MTE coincides with the notion of Wardrop equilibrium.

Our final remark is that Proposition 1 establishes that an MTE gives us the optimal flow v^* in terms of an implicit equation. To see this, we note that at each node $i \neq d$, v^* can be rewritten as

$$v_a^* = x_i \frac{e^{\beta_i z_a}}{\sum_{b \in A_i^+} e^{\beta_i z_b}} = x_i \frac{e^{\beta_i (R - p_a - l_a(v_a^*) + \tau_{j_a})}}{\sum_{b \in A_i^+} e^{\beta_i (R - p_b - l_a(v_b^*) + \tau_{j_b})}}, \quad \forall a \in A_i^+. \tag{8}$$

Expression (8) makes clear the fact that the optimal solution v^* is in terms of an implicit equation.[17]

The uniqueness of an MTE allows us to define v^* as $v^* \equiv D(p)$, where $D(p) = (D_a(p))_{a \in A}$. We called $D(p)$ the *demand function* for the traffic problem. By the Maximum Theorem, $D(p)$ is a continuous function of p. Thus, the following corollary is straightforward.

Corollary 1. *Let $D(p)$ be an MTE. Then the profit function (2) is a continuous function of p. Furthermore, (2) can be written as*

$$\pi_a(p) = p_a \left(x_i \frac{e^{\beta_i (R - p_a - l_a(D_a(p)) + \tau_{j_a})}}{\sum_{b \in A_i^+} e^{\beta_i (R - p_b - l_b(D_b(p)) + \tau_{j_a})}} \right), \quad \forall a \in A_i^+, \ i \neq d, \tag{9}$$

where x_i satisfies (5).

The previous corollary is important because it illustrates two important features of the profit functions represented by $\pi_a(\cdot)$. First, it explicitly shows how the congestion levels affects the shape of the profit functions. In fact, we shall see that this feature plays a central role in establishing the existence and uniqueness of an OE.

The second feature is that a firm setting prices will worry either about its link being excluded from the optimal path, or, when retained in the optimal path, about the reduction on the overall demand for the path. The prices p_a and continuation values τ capture these effects in our model. We shall refer to these effects as the *path effect* and the *demand effect*, respectively.

3. Oligopoly pricing: Existence and uniqueness of a symmetric price equilibrium

In this section we begin the study of a price equilibrium by considering a symmetric model. The general definition of an oligopoly price equilibrium is the following.

Definition 2. A pair $(p^{OE}, D(p^{OE}))$ is a pure strategy Oligopoly Price Equilibrium (OE) if for all $a \in A$

$$p_a^{OE} \in \arg \max_{p_a \in [0,R]} \left\{ \pi_a\left(D_a\left(p_a, p_{-a}^{OE}\right)\right) \right\}, \quad \forall p_{-a}^{OE}, \tag{10}$$

where $D(p_a, p_{-a}^{OE})$ is the MTE for the price vector (p_a, p_{-a}^{OE}).

Definition 2 is the standard notion of a sub-game perfect Nash equilibrium applied to the pricing game under study and it does not impose any restriction on the network topology.

We now specialize Definition 2 to the case of a symmetric model. In particular, in the symmetric model we assume that the congestion at each link a is captured by the same latency function, namely $l(\cdot)$. We assume $\beta_i = \beta$ and $g_i = g$ for all $i \neq d$. Furthermore, we assume that any pair of nodes is connected by at least two links, where the number of available links is denoted by n_i for any node $i \neq d$.

Combining Definition 2 with the symmetry in the model, we can define the notion of a symmetric OE as follows.

Definition 3. We say that a pure strategy OE given by $(p^{OE}, D(p^{OE}))$ is symmetric if and only if for all $i \neq d$

$$p_a^{OE} = p_{n_i}^{OE}, \quad \forall a \in A_i^+,$$

with $n_i = |A_i^+|$.

Definition 3 just says that at each node $i \neq d$, firms set the same price, which depends on the number of firms on that node. In fact, Proposition 5 and Corollary 2 in Appendix A show that for a symmetric OE, the prices and profits are decreasing in the number of firms. These results use that $l(\cdot)$ is a convex function.

Our first result is the existence and uniqueness of a symmetric OE. Formally we get:

Theorem 1. *There exists a unique symmetric OE.*

[17] The derivation of (8) is as follows. Given the flow x_i at node i, the probability of choosing link $a \in A_i^+$ is given by $\mathbb{P}(u_a + \varphi_{j_a}(z) + \epsilon_a > u_b + \varphi_{j_b}(z) + \epsilon_b, \forall b \in A_i^+)$. Using Assumption 2 combined with the expression for u_a, u_b, φ·s we get (8).

The proof of Theorem 1 is based on checking the technical conditions on Assumption 3 in Section 4 below. In particular, Assumption 3 is a condition on the latency functions which guarantees that the profit functions are concave, so that the Kakutani fixed point Theorem can be invoked. Interestingly, for the symmetric case such technical conditions are automatically satisfied, and no further assumptions on the class of latency functions are required to guarantee the existence and uniqueness of a symmetric OE.[18]

3.1. Welfare analysis and entry decisions

Provided the existence and uniqueness of a symmetric OE, we ask the following question: Under free entry, *will the number of firms be socially optimal*? We interpret entry decisions as new links in the network. Considering a fixed set of nodes, whenever a firm enters the market, the network topology changes. In this environment, a social planner will look for the optimal number of links connecting the nodes, i.e., he will look for the socially optimal design of the network. Our main result in this section establishes that under free entry, and given a fixed cost, the number of firms that enter the market is larger than the socially optimal, i.e., there is excess entry in this setting. The excess entry result is due to a new firm entering the market reduces the demand and prices of the existing firms in the network. This phenomenon is known as "*The business-stealing effect*" (Mankiw and Whinston, 1986). Intuitively, the business-stealing by a marginal entrant drives a wedge between the entrant's evaluation of the desirability of his entry and the social planner's, generating the discrepancy between n^E and n^S.

We introduce the presence of a social welfare measure, which is given by the sum of *firms' surplus* and *users' surplus*.

Definition 4. Let $(p^{OE}, D(p^{OE}))$ be a pure strategy OE. We define the aggregate welfare as

$$\mathbb{W}\big(p^{OE}, D\big(p^{OE}\big)\big) \equiv \sum_{i\neq d} \mathbb{W}_i\big(p^{OE}, D\big(p^{OE}\big)\big), \tag{11}$$

where $\mathbb{W}_i(p^{OE}, D(p^{OE})) \equiv \sum_{a\in A_i^+} \pi_a(D_a(p^{OE})) + x_i\mathbb{E}(\max_{a\in A_i^+}\{z_a+\epsilon_a\})$ is the welfare at node $i\neq d$ with x_i satisfying the flow constraint (5) and $z_a = u_a + \tau_{j_a}$.

In Definition 4, the term $\sum_{a\in A_i^+} \pi_a(D_a(p^{OE}))$ represents firms' surplus while $x_i\mathbb{E}(\max_{a\in A_i^+}\{z_a+\epsilon_a\})$ represents users' surplus. Thanks to Assumption 2, formula (11) can be written in a closed form, where

$$\mathbb{W}_i\big(p^{OE}, D\big(p^{OE}\big)\big) = \sum_{a\in A_i^+} \pi_a\big(D_a\big(p^{OE}\big)\big) + \frac{x_i}{\beta}\log\bigg(\sum_{a\in A_i^+} e^{\beta z_a}\bigg), \quad \forall i\neq d. \tag{12}$$

We note that Definition 4 explicitly uses the Markovian structure of the model. In fact, the aggregate welfare is just the sum of welfare at each node $i\neq d$, which follows from the recursive structure on users' decisions.

In order to analyze entry decisions, we consider a fixed entry cost, which is denominated as *sunk cost* and is denoted by F.[19] Thus, given a price vector p and the sunk cost F, the profit functions may be written as:

$$\pi_a\big(D_a(p)\big) = p_a D_a(p) - F, \quad \forall a\in A. \tag{13}$$

Using this simple framework, we are able to answer whether the market will provide the optimal number of firms or not. In particular, we compare the solution obtained by a social planner with the solution obtained by the market. The social planner maximizes the social welfare choosing the optimal number of firms. Formally, the planner solves the following optimization problem:

$$\max_n\big\{\mathbb{W}\big(p_n^{OE}, D\big(p_n^{OE}\big)\big)\big\} = \max_n\bigg\{\sum_{i\neq d}\bigg(\sum_{a\in A_i^+}\pi_a\big(D_a\big(p_n^{OE}\big)\big) + \frac{x_i}{\beta}\log\bigg(\sum_{a\in A_i^+} e^{\beta z_a}\bigg)\bigg)\bigg\}, \tag{14}$$

where $(p_n^{OE}, D(p_n^{OE}))$ denotes a symmetric equilibrium. Due to symmetry, denote $\mathbb{W}(p_n^{OE}, D(p_n^{OE}))$ as $\mathbb{W}(n)$, so that expression (14) may be rewritten as:

$$\max_n\big\{\mathbb{W}(n)\big\} = \max_n\bigg\{\sum_{i\neq d}\bigg(\frac{x_i}{\beta}\log(n_i) - x_i l(x_i/n_i) - n_i F\bigg)\bigg\}, \tag{15}$$

where the last expression is obtained due to the symmetry of the problem.

[18] Intuitively, Theorem 1 establishes that the demand system induced by the MTE is strictly concave, which implies that the firms' best response map is convex. See Section 4 for the details of the derivation of the strict concavity of the firms' profit functions in the general case.

[19] We can interpret the term F as the cost that a firm must pay to participate in the market.

We note that expression (15) is a strictly concave function in $n = (n_i)_{i \neq d}$, so the first order conditions are necessary and sufficient for finding the socially optimal number of firms $n^S = (n_i^S)_{i \neq d}$ for the whole network. On the other hand, the equilibrium condition for firms entering at each node $i \neq d$, is given by the zero profit condition:

$$\forall a \in A_i^+, \quad \pi_a\big(p_{n_i}^{OE}\big) = 0. \tag{16}$$

Thus solving Eqs. (16) we get the equilibrium number of firms n_i^E, where for the whole network the number of firms is given by $n^E = (n_i^E)_{i \neq d}$. We remark that thanks to the convexity of the latency functions, the system of Eqs. (16) has a unique solution.

Previous setting allows us to formalize our initial question in the following way: What is the relationship between n^E and n^S?

The following theorem gives an answer to this question.

Theorem 2. *In the symmetric congestion pricing game $n^E > n^S$.*

The proof of Theorem 2 relies on finding n^S and n^E solving (15) and (16) respectively. Once n^E and n^S have been found, we proceed to compare them concluding that $n^E > n^S$.

It is worth noting two underlying aspects in Theorem 2. First, as we said before, our result is based in the idea of the *business-stealing* effect. In fact, in Appendix A we show that prices and profits are decreasing in the number of firms (Proposition 5 and Corollary 2 respectively). Thus an entering firm does not internalize such an effect, while the social planner's solution considers this externality. Second, the proof of Theorem 2 also relies on the assumption of convexity of the latency function $l(\cdot)$. If the latency functions are not convex, then the result in Theorem 2 no longer holds.

Theorem 2 generalizes the results in Anderson et al. (1995) and Mankiw and Whinston (1986) to the case of a congestion pricing game with a general network topology. Similarly, Theorem 2 generalizes the result in Weintraub et al. (2010) to the case of a general network.

4. Existence and uniqueness of an OE: The general case

The goal of this section is to establish the existence and uniqueness of an OE for a general class of latency functions. In our study, we shall restrict attention to an OE such that at any node $i \neq d$, the users' utilities satisfy[20]

$$R - p_a - l_a\big(D_a(p)\big) + \tau_{j_a} = R - p_b - l_b\big(D_b(p)\big) + \tau_{j_b}, \quad \text{for all } a \neq b \in A_i^+.$$

This condition makes explicit the fact that any firm a setting prices takes into account the path effect and the demand effect of its price setting behavior. Moreover, restricting our attention to this class of equilibrium has the advantages of its simplicity and comparability with previous results in the literature.[21]

4.1. Existence

In order to study the existence of an OE, we exploit the Markovian structure of the users' decisions combined with the assumption that every link is owned by a different firm. In fact, due to the Markovian structure, we can decompose the pricing problem for the whole network into a collection of pricing problems at each node $i \neq d$. Thus the firms competing at node $i \neq d$ set their prices taking as given the flow of users starting at node i, and the prices set by firms in different nodes.

Using this structure, we study the problem of existence through the application of Kakutani's fixed point Theorem. In order to apply Kakutani's result, we need to check that the best response map is non-empty, upper semi-continuous, and convex valued. For the pricing game under analysis, the fact that the best response is not empty and upper semi-continuous follows a straightforward application of the Maximum Theorem. However, the best response map is not convex valued, which makes the application of Kakutani's Theorem problematic. In this paper we provide a specific condition in order to guarantee the convexity of the best response. The condition depends on the latency functions and it is automatically satisfied in the symmetric case we analyzed in Section 2.

Formally, given any price vector $p \geqslant 0$, we define firm a's best response map $B_{ia}(p_{-a})$ as follows: for all $i \neq d$, $a \in A_i^+$,

$$B_{ia}(p_{-a}) = \underset{p_a \in [0,R]}{\arg\max} \big\{\pi_a\big(D_a(p_a, p_{-a})\big)\big\}.$$

[20] Recall that the index j_a denotes that node j has been reached using link a.

[21] As we said before, most of the existent results on pricing in congested networks make use of the concept of Wardrop equilibrium.

To study the convexity of $B(\cdot)$, we analyze the concavity of the profit functions π_a. Recall that for each firm a the profit function is given by $\pi_a(D_a(p)) = p_a D_a(p)$. In order to establish the concavity of the π_a we note that for all p^{OE}, profit maximization implies that the following optimality condition must hold

$$\forall a \in A \quad \frac{\partial \pi_a(D_a(p^{OE}))}{\partial p_a} = D_a\big(p^{OE}\big) + p_a^{OE}\frac{\partial D_a(p^{OE})}{\partial p_a} = 0. \tag{17}$$

Using (17), we get

$$\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a^2} = 2\frac{\partial D_a(p^{OE})}{\partial p_a} + p_a^{OE}\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2}. \tag{18}$$

The profit function is strictly concave if and only if expression (18) is negative. In particular, for the case of uncongested markets, and under the assumption that the distribution of the ϵ_a is double exponential, expression (18) is negative, and the concavity of the profit function holds. Thus, for the case of uncongested markets, the existence of an OE follows directly. However, we know from (8) that the presence of congestion effects implies that the demand function is defined in implicit terms, so when computing $\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a^2}$ we must take into account this feature. From (18) it follows that the concavity depends on $\frac{\partial D_a(p^{OE})}{\partial p_a}$ and $\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2}$. In Appendix A we show that $\frac{\partial D_a(p^{OE})}{\partial p_a} < 0$ and $\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2}$ can be written as

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} = -\frac{1}{D_a(p^{OE})}\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^2 \big[K_{ia}\big(p^{OE}\big) - 2\big],$$

where $K_{ia}(p^{OE})$ is the term that takes into account the effect of the latency functions $l_a(\cdot)$ into the sign (and shape) of $\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2}$. Using the previous expression combined with (17), we can rewrite (18) as[22]

$$\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a^2} = \frac{\partial D_a(p^{OE})}{\partial p_a} K_{ia}\big(p^{OE}\big). \tag{19}$$

Eq. (19) shows that concavity of the profit function relies on understanding the terms $K_{ia}(p^{OE})$. In particular, the profit function will be concave if $K_{ia}(p^{OE}) > 0$.

The derivation of the $K_{ia}(p^{OE})$'s is quite involved, and we refer the reader to Appendix A for details. We define the $K_{ia}(p^{OE})$ in the following way; for all $i \neq d$, $a \in A_i^+$

$$K_{ia}\big(p^{OE}\big) = 2 + \frac{\beta_i D_a(p^{OE})}{J_{ia}}\left(\Omega_{ia}\big(p^{OE}\big) + (1 - 2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1}\right), \tag{20}$$

where $\mathbb{P}_a = \frac{e^{\beta_i z_a}}{\sum_{b \in A_i^+} e^{\beta_i z_b}}$, and $J_{ia} = 1 + \beta_i D_a(p^{OE})\big((1 - \mathbb{P}_a) l_a'(D_a(p^{OE})) + \sum_{b \neq a} \frac{\mathbb{P}_b l_b'(D_b(p^{OE}))}{(n_i - 1)}\big)$.

The term $\Omega_{ia}(p^{OE})$, can be decomposed as

$$\Omega_{ia}\big(p^{OE}\big) = \underbrace{\text{Firm } a\text{'s congestion}}_{\mathcal{C}_a(p^{OE})} + \underbrace{\text{Competitors' congestion}}_{\mathcal{C}_{-a}(p^{OE})},$$

with $\mathcal{C}_a(p^{OE}) = (1 - 2\mathbb{P}_a) l_a'(D_a(p^{OE})) + D_a(p^{OE})(1 - \mathbb{P}_a) l_a''(D_a(p^{OE}))$, and $\mathcal{C}_{-a}(p^{OE}) = \sum_{b \neq a} l_b'(D_b(p^{OE}))\big(\frac{(n_i - 1)\mathbb{P}_b - \mathbb{P}_a}{(n_i - 1)^2}\big) - \sum_{b \neq a} \frac{D_a(p^{OE})\mathbb{P}_b l_b''(D_b(p^{OE}))}{(n_i - 1)^2}$.

The component $\mathcal{C}_a(p^{OE})$ shows the effect of firm a's latency function l_a, while $\mathcal{C}_{-a}(p^{OE})$ can be viewed as the average effect of firm a's competitors' latency functions l_b, with $b \neq a$. Thanks to this decomposition, the term $\Omega_{ia}(p^{OE})$ captures all relevant information to determine the concavity of the profit function.[23] First, we note that $\Omega_{ia}(p^{OE})$ depends on the l_a' and l_a''. By assumption, we know that the latency functions l_a are strictly increasing and convex, so it follows that $\mathcal{C}_a(p^{OE})$ is strictly positive. For the case of $\mathcal{C}_{-a}(p^{OE})$, a more careful analysis must be carried out.

In fact, in Appendix A we analyze how $\mathcal{C}_{-a}(p^{OE})$ determines the sign of $K_{ia}(p^{OE})$ through its effect on $\Omega_{ia}(p^{OE})$. The main message from that analysis is that for highly congested networks, the sign of $K_{ia}(p^{OE})$ can be negative, which implies that the condition of strict concavity of the profit function can be violated.

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22 The derivation of (19) is the following. From the first order condition, we know that $D_a(p^{OE}) = -p_a^{OE}\frac{\partial D_a(p^{OE})}{\partial p_a}$. Using this fact we note that $p_a^{OE}\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} = 2\frac{\partial D_a(p^{OE})}{\partial p_a}$. Then replacing the last expression into $\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a^2}$ the expression follows at once.

23 Throughout the analysis, and without loss of generality, we shall assume that at each node $i \neq d$ the equilibrium probabilities satisfy $\sum_{b \neq a} \mathbb{P}_b \geqslant \mathbb{P}_a$ for all $a \in A_i^+$.

4.2. Uniqueness

Similar to the study of existence, an explicit condition can be derived to analyze the uniqueness of an OE. In particular, we study the uniqueness based on the *dominant diagonal property* (see Vives, 2001, Ch. 2), which establishes that the equilibrium is unique if the following condition holds:

$$\forall i \neq d,\ \forall a, b, \in A_i^+; \quad -\sum_{b \neq a} \frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a \partial p_b} \left[\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a^2} \right]^{-1} < 1.$$

The previous condition shows that the uniqueness depends on the positivity of $\frac{\partial^2 \pi_a(D_a(p^{OE}))}{\partial p_a \partial p_b}$, which can be written as

$$\frac{\partial^2 \pi(D_a(p^{OE}))}{\partial p_a \partial p_b} = \frac{\partial D_a(p^{OE})}{\partial p_b} + p_a^{OE} \frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b}. \tag{21}$$

In Appendix A, Lemma 1 shows that $\frac{\partial D_a(p^{OE})}{\partial p_a^{OE}} > 0$, and Lemma 3 shows that $\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b}$ can be written as

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b} = -\frac{1}{D_a(p^{OE})} \left[\frac{\partial D_a(p^{OE})}{\partial p_a} \frac{\partial D_a(p^{OE})}{\partial p_b} \right] \left[\bar{K}_{iab}(p^{OE}) - 1 \right],$$

where $\bar{K}_{iab}(p^{OE})$ is the term that takes into account the effect of the $l_a'(\cdot)$s and $l_a''(\cdot)$s in the sign (and shape) of $\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b}$. Using the previous expression, we rewrite (21) as

$$\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a \partial p_b} = \frac{\partial D_a(p^{OE})}{\partial p_b} \bar{K}_{iab}(p^{OE}). \tag{22}$$

Thus expression (22) is positive if $\bar{K}_{iab}(p^{OE})$ is strictly positive. For uncongested markets, and under Assumptions 1 and 2, expression (22) is strictly positive.

Similar to the case of $K_{ia}(p^{OE})$, the derivation of $\bar{K}_{iab}(p^{OE})$ is involved and we refer the reader to Appendix A. These terms are defined as follows; for all $i \neq d$, $a, b \in A_i^+$

$$\bar{K}_{iab}(p^{OE}) = 1 + \frac{\beta_i D_a(p^{OE})}{J_{ia}} \left(\Omega_{ia}(p^{OE}) - \left[\frac{\partial D_a(p^{OE})}{\partial p_b} \right]^{-1} \left[\mathbb{P}_b - \frac{\mathbb{P}_a}{(n_i - 1)} \right] \right) \tag{23}$$

with Ω_{ia}, $\mathbb{P}_a$, $\mathbb{P}_b$, and J_{ia} defined as before. Furthermore, we note that points (1) and (2) made for the $K_{ia}(p^{OE})$'s also apply to the case of the $\bar{K}_{iab}(p^{OE})$'s.

From the previous analysis, it is clear that the existence and uniqueness of an OE follows if $K_{ia}(p^{OE}) > 0$ and $\bar{K}_{iab}(p^{OE}) > 0$. Formally

Assumption 3. For any node $i \neq d$, and for all $a, b \in A_i^+$, let $K_{ia}(p^{OE})$ and $\bar{K}_{iab}(p^{OE})$ be given by expressions (20) and (23) respectively. The latency functions are such that: $K_{ia}(p^{OE}) > 0$ and $\bar{K}_{iab}(p^{OE}) > 0$ for all p^{OE}.

We point out that Assumption 3 is not vacuous. The following examples illustrate how Assumption 3 can apply to two classes of latency functions.

Example 2 *(Linear class).* Our first example is the class of linear latency functions. This class is given by the functions $l_a(D_a(p)) = \delta_a D_a(p)$, with $\delta_a > 0$ for all $a \in A$. A straightforward computing shows that for this class of functions $K_{ia}(p^{OE})$ is strictly positive. The reason for this is because $l_a''(D_a(p)) = 0$ for all $a \in A$. This implies that the negative term in $\mathcal{C}_{-a}$ does not play any role. Thus for the case of linear latency functions, the profit function is concave in its own price. Furthermore, it is easy to see that for the linear case $\bar{K}_{iab} > 0$. Thus, for the case of linear latency functions, the Assumption 3 is satisfied.

Example 3 *(Load balancing class).* Let us consider the class of latency functions given by $l_a(D_a(p)) = (\mu_a - D_a(p))^{-1}$ with $\mu_a > D_a(p)$ for all $p \geqslant 0$ and $a \in A$. The parameter $\mu_a > 0$ represents the capacity of each link a. As we said before, this class of functions is the leading case in the context of queueing games (see Hassin and Haviv, 2006). This class is strictly increasing and strictly convex, where $l_a'(D_a(p)) = l_a^2(D_a(p))$ and $l_a''(D_a(p)) = 2l_a^3(D_a(p))$. From this latter property, it follows that $l_a''(D_a(p)) \to \infty$ as $D_a(p) \to \mu_a$, which implies that $\mathcal{C}_{-a}(p^{OE}) + (1 - 2\mathbb{P}_a)[\frac{\partial D_a(p^{OE})}{\partial p_a}]^{-1}$ can be arbitrarily large and negative if some of firm a's competitors are operating very close to their link capacities. This behavior can make $K_{ia}(p^{OE})$ negative, and as a consequence, the concavity of the profit function will fail. The intuition for this observation is that for highly congested networks operating close to the capacity of their links, an OE may not exist.[24] Using the converse argument, it can be established that if at each node $i \neq d$ the μ_as are such that the strict inequality $\sum_{a\in A_i^+} \mu_a > \sum_{a\in A_i^+} D_a(p)$ holds, then the conditions of Assumption 3 will apply, and the profit functions will be concave. A similar reasoning can be applied to analyze the sign of $\bar{K}_{iab}(p^{OE})$.

From an applied point of view, we note that the conditions in Assumption 3 provide information for the design of large scale simulation exercises having a unique OE.

Finally, the main result can be formally written as:

Theorem 3. *Suppose that Assumptions 2 and 3 hold. Then, there exists a unique OE.*

Three aspects are worth emphasizing with regards to Theorem 3. First, we note that Theorem 3 is a generalization of the results available in the literature of oligopoly pricing in congested markets. Indeed, practically all environments considered in the literature satisfy Assumption 3. Furthermore, for the case of networked markets without congestion effects, Theorem 3 also applies. In fact, Assumption 3 trivially holds. Thus Theorem 3 can also be viewed as an extension of the results of existence and uniqueness of an OE in the standard oligopoly theory. Second, Theorem 3 establishes the *existence* and *uniqueness* of an OE for a general class of network topologies. Finally, we stress that Theorem 3 considers certain heterogeneity among the random variables because of the different β_i at each node, i.e. the result applies to the case where at every node the population heterogeneity is different.

In addition to the existence and uniqueness of an OE, we provide an explicit characterization of the equilibrium price vector p^{OE}.

Proposition 2. *Let $(p^{OE}, D(p^{OE}))$ be a pure strategy OE. Then, for all $i \neq d$, and $a \in A_i^+$:*

$$p_a^{OE} = \frac{1}{\beta_i(1-\mathbb{P}_a)} + D_a\big(p^{OE}\big)\left[(1-\mathbb{P}_a)l_a'\big(D_a\big(p^{OE}\big)\big) + \frac{\sum_{b\neq a}\mathbb{P}_b l_b'((D_b(p^{OE}))}{n_i - 1}\right],$$

where $n_i = |A_i^+|$, and $\mathbb{P}_a = \frac{e^{\beta_i z_a}}{\sum_{b\in A_i^+} e^{\beta_i z_b}}$ for all $a \in A_i^+$.

Proposition 2 establishes that the equilibrium price vector p^{OE} can be expressed as a function of two components. The first component is due to the fact that p^{OE} depends on the dispersion parameters β_i. In particular, at each node $i \neq d$, the equilibrium prices include the terms $\frac{1}{\beta_i(1-\mathbb{P}_a)}$. This contrasts with the expression that we would get if the notion of Wardrop equilibrium were considered. The reason for this discrepancy is because our approach allows for heterogeneity within users, whereas Wardrop equilibrium works for a homogeneous population of users. However, as $\beta_i \to \infty$ for all $i \neq d$, the term $\frac{1}{\beta_i(1-\mathbb{P}_a)}$ goes to zero, which implies that equilibrium prices resemble the ones obtained when Wardrop equilibrium is considered as the solution concept.[25]

The second component is $D_a(p^{OE})[(1-\mathbb{P}_a)l_a'(D_a(p^{OE})) + \frac{\sum_{b\neq a}\mathbb{P}_b l_b'(D_b(p^{OE}))}{n_i - 1}]$. From the previous expression, it is easy to see that $D_a(p^{OE})(1-\mathbb{P}_a)l_a'(D_a(p^{OE}))$ is the Pigouvian pricing, which must be charged by firms such that users internalize the congestion externality. Regarding the term $D_a(p^{OE})\frac{\sum_{b\neq a}\mathbb{P}_b l_b'(D_b(p^{OE}))}{n_i - 1}$, it has the interpretation of *an extra markup* due to oligopolistic competition among firms at each node $i \neq d$.

Summarizing, the equilibrium price vector p^{OE} can be viewed as the sum of three factors: *Heterogeneity, Pigouvian pricing, and extra markup*.

Finally, we note that Proposition 2 shows that equilibrium prices solve a fixed point equation, where for each firm a the equilibrium price p_a^{OE} depends on two factors: the prices of its competitors and the continuation values τ_{j_a} associated to its link. These two elements make explicit that a firm setting prices considers the path and demand effects. The previous effects combined with the heterogeneity in users (through the β_i) turn out to be new elements in the study of price competition in congested networks.

5. Conclusion and final remarks

In this paper we have studied the problems of free entry and welfare, and the existence and uniqueness of an OE in congested markets for a quite general class of networks.

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[24] A similar observation in the context of load balancing games can be found in Anselmi et al. (2011). However, they do not provide conditions to study the existence of an OE. In particular, expression (20) formalizes their intuition.

[25] Recall that at each node $i \neq d$, the variance of the random variable $\{\epsilon_a\}_{a\in A_i^+}$ is given by $\mathbb{V}(\epsilon_a) = \frac{\pi}{\beta_i\sqrt{6}}$. Then, $\beta_i \to \infty$ implies that the variance goes to zero, meaning that utility within the population is homogeneous. This latter interpretation allows us to compare our result in Proposition 2 with the prices that would be obtained using Wardrop equilibria as equilibrium concept for users' problem.

In particular, we provided conditions under which an OE exists and is unique in a general class of environments, encompassing many setting studied in the literature. These results allow to us to inspect the welfare properties of congested networks under free entry. To the best of our knowledge, our paper is the first to establish the result of excess entry for the case of congestion pricing games in a general network. The closest result to ours is the recent paper by Weintraub et al. (2010), who consider a simple network having a single origin–destination pair with a collection of parallel links. Consequently, we think that our result may provide insights regarding the design of optimal networks subject to congestion effects.

Finally, the introduction of random utility models to the study of pricing in congested networks opens the possibility of carrying out two interesting exercises. The first exercise is related to the evaluation of changes in users' welfare, using the demand function generated at every node. Concretely, and given the result in Theorem 3, we can evaluate the impact in users' welfare of different pricing policies. From an econometric viewpoint, the second exercise is related to the estimation of a congestion pricing game, mimicking the empirical I.O. literature.

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Appendix A. Proofs

We begin this appendix with the proof of the existence and uniqueness of an MTE.

Proof of Proposition 1. For a given $p \geqslant 0$, let v^* be an MTE. Since (P) is a strictly concave program with respect to v, it suffices to check that $v^* \in V$ is a stationary point of the Lagrangian:

$$\mathcal{L} = \sum_{a\in A}(R - p_a)v_a - \sum_{a\in A}\int_0^{v_a} l_a(s)\,ds - \chi(v) + \sum_{i\neq d}\mu_i\Big[g_i + \sum_{a\in A_i^-} v_a - \sum_{a\in A_i^+} v_a\Big] - \sum_{a\in A}\lambda_a v_a.$$

The multipliers $\mu_i \in \mathbb{R}$, and $\lambda_a \geqslant 0$ correspond to (1) and $v_a \geqslant 0$ respectively, and stationary amounts to $R - p_a - l_a(v_a^*) = u_a^*$, and $\zeta = \nabla\chi(v)$ where $\zeta = \mu_{i_a} + u_a^* - \mu_{j_a} - \lambda_a$. For the multipliers λ_a, we simply set $\lambda_a = 0$. To check the last condition take $\mu_i = \tau_i(u^*)$. Combining the (1) and (5) we get

$$v_a = \frac{\partial\varphi_i}{\partial z_a}(z)\sum_{a\in A_i^+} v_a = \left(\frac{e^{\beta_i z_a}}{\sum_{b\in A_i^+} e^{\beta_i z_b}}\right)\sum_{a\in A_i^+} v_a$$

which shows that z is a optimal solution for $\chi(v)$ and therefore setting $g_a = \varphi_{i_a}(z) - z_a$ we get $g = \nabla\chi(v)$. Since $\varphi_{i_a}(z) = \tau_i$ and $z_a = u_a^* + \tau_{j_a}$ we deduce that $g = \zeta = \nabla\chi(v)$ as required. $\square$

We note that Proposition 1 can be extended to the case of dependent random variables at each node $i \neq d$. The main difference in the proof would be that the expression $v_a = \frac{\partial\varphi_i}{\partial z_a}(z)\sum_{a\in A_i^+} v_a$ does not have a closed form expression. In addition, we would need to assume that the density of the random variables ϵ_a is twice differentiable.

A.1. Quasi-concavity of the profit function

In this section we derive the conditions for the quasi-concavity of the profit function $\pi_a(\cdot)$ for all $a \in A$. In particular, we find expressions for $\frac{\partial^2\pi(p_a, p_{-a}^{OE})}{\partial p_a^2}$ and $\frac{\partial^2\pi(p_a, p_{-a}^{OE})}{\partial p_a\partial p_b}$, for $a \neq b$. Our first result is the characterization of $\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}$ and $\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_b}$ for $b \neq a$.

Lemma 1. *Let $(p^{OE}, D(p^{OE}))$ be a pure strategy OE. Then, $\forall i \neq d$, $a, b \in A_i^+$, with $a \neq b$ we get:*

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} = \frac{-\beta_i D_a(p_a, p_{-a}^{OE})(1 - \mathbb{P}_a)}{J_{ia}}, \tag{24}$$

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_b} = \frac{\beta_i D_a(p_a, p_{-a}^{OE})\mathbb{P}_b}{J_{ia}}, \tag{25}$$

where

$$J_{ia} \equiv 1 + \beta_i D_a\big(p_a, p_{-a}^{OE}\big)(1 - \mathbb{P}_a)\left[l_a'\big(D_a\big(p_a, p_{-a}^{OE}\big)\big) + \frac{\sum_{b \neq a} q_b l_b'(D_b(p_a, p_{-a}^{OE}))}{(n_i - 1)}\right],$$

with $n_i = |A_i^+|$, $\mathbb{P}_a = \frac{e^{\beta_i z_a}}{\sum_{b \in A_i^+} e^{\beta_i z_b}}$, *and* $q_b = \frac{\mathbb{P}_b}{1-\mathbb{P}_a}$.

Proof. Let us fix a node $i \neq d$. Considering $D_a(p_a, p_{-a}^{OE})$ and taking partial derivative with respect to p_a we get

$$\begin{aligned}\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} &= -\beta_i D_a\big(p_a, p_{-a}^{OE}\big)(1 - \mathbb{P}_a)\left[1 + l_a'\big(D_a\big(p_a, p_{-a}^{OE}\big)\big)\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}\right] \\ &\quad + \beta_i D_a\big(p_a, p_{-a}^{OE}\big)\sum_{b \neq a}\mathbb{P}_b l_b'\big(D_b\big(p_a, p_{-a}^{OE}\big)\big)\frac{\partial D_b(p_a, p_{-a}^{OE})}{\partial p_a}.\end{aligned}$$

On the other hand, for the entering flow x_i, we know that it must satisfy $\sum_{b \in A_i^+} D_b(p_a, p_{-a}^{OE}) = x_i$. Using the fact that p^{OE} is a pure strategy OE it follows that

$$\begin{aligned}\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} &= -\beta_i D_a\big(p_a, p_{-a}^{OE}\big)(1 - \mathbb{P}_a)\left[1 + l_a'\big(D_a\big(p_a, p_{-a}^{OE}\big)\big)\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}\right] \\ &\quad - \frac{\beta_i D_a(p_a, p_{-a}^{OE})}{n_i - 1}\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}\sum_{b \neq a}\mathbb{P}_b l_b'\big(D_b\big(p_a, p_{-a}^{OE}\big)\big).\end{aligned}$$

Solving for $\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}$ we get:

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} = \frac{-\beta_i D_a(p_a, p_{-a}^{OE})(1 - \mathbb{P}_a)}{1 + \beta_i D_a(p_a, p_{-a}^{OE})(1 - \mathbb{P}_a)\big[l_a'(D_a(p_a, p_{-a}^{OE})) + \sum_{b \neq a}\frac{\mathbb{P}_b}{1-\mathbb{P}_a}\frac{l_b'(D_b(p_a, p_{-a}^{OE}))}{n_i - 1}\big]}.$$

Finally, using the definition for q_b (with $b \neq a$) and J_{ia}, we find:

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} = \frac{-\beta_i D_a(p_a, p_{-a}^{OE})(1 - \mathbb{P}_a)}{J_{ia}}.$$

As the previous analysis holds for any node $i \neq d$, the conclusion follows. For the case of $\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_b}$, the same logic yields

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_b} = \frac{\beta_i D_a(p_a, p_{-a}^{OE})\mathbb{P}_b}{J_{ia}}, \quad \forall b \neq a \in A. \qquad \square$$

When there is no congestion at the network we get $l_a'(x_a) = 0$ for all x_a and for all $a \in A$, so that we find $\frac{\partial D_a(p_a, p_a^{OE})}{\partial p_a} = -\beta_i D_a(p_a, p_{-a}^{OE})(1 - \mathbb{P}_a)$, and $\frac{\partial D_a(p_a, p_a^{OE})}{\partial p_b} = \beta_i D_a(p_a, p_{-a}^{OE})\mathbb{P}_b$ for $b \neq a$. Thus, Lemma 1 can be viewed as a generalization of the demand behavior to the case of oligopoly competition in congested markets.

We now introduce the terms $K_{ia}(p^{OE})$ and $\bar{K}_{iab}(p^{OE})$ as follows: For all $i \neq d$, and $a \in A_i^+$, define

$$K_{ia}\big(p^{OE}\big) \equiv 2 + \frac{\beta_i D_a}{J_{ia}}\Omega_{ia} + \frac{\beta_i D_a}{J_{ia}}\left[\frac{\partial D_a}{\partial p_a}\right]^{-1}(1 - 2\mathbb{P}_a),$$

where $\Omega_{ia} = \big[(1 - 2\mathbb{P}_a)l_a' + \sum_{b \neq a} l_b'(\frac{(n_i-1)\mathbb{P}_b - \mathbb{P}_a}{(n_i-1)^2}) + D_a(1 - \mathbb{P}_a)l_a'' - \sum_{b \neq a}\frac{D_a\mathbb{P}_b l_b''}{(n_i-1)^2}\big]$, $\mathbb{P}_a = \frac{e^{\beta_i z_a}}{\sum_{b \in A_i^+} e^{\beta_i z_b}}$, $J_{ia} \equiv 1 + \beta_i\big(D_a(1 - \mathbb{P}_a)l_a' + \sum_{b \neq a}\frac{D_a\mathbb{P}_b l_b'}{(n_i-1)}\big)$, $D_a \equiv D_a(p^{OE})$, $l_a' \equiv l_a'(p^{OE})$, and $l_a'' \equiv l_a''(p^{OE})$ for all $a \in A$.

Similarly, we define $\bar{K}_{iab}(p^{OE})$ as:

$$\bar{K}_{iab}\big(p^{OE}\big) = 1 + \frac{\beta_i D_a}{J_{ia}}\Omega_{ia} - \frac{\beta_i D_a}{J_{ia}}\left[\frac{\partial D_a}{\partial p_b}\right]^{-1}\left(\frac{(n_i - 1)\mathbb{P}_b - \mathbb{P}_a}{(n_i - 1)}\right),$$

with Ω_{ia}, $\mathbb{P}_a$, J_{ia}, D_a, l_a', and l_a'' defined as before.

As we pointed out in the main text, the terms $K_{ia}(p^{OE})$ and $\bar{K}_{iab}(p^{OE})$ can be viewed as technical conditions on the class of latency functions.

The term $K_{ia}(p^{OE})$ is derived as follows. From Lemma 1 we know that $\frac{\partial D_a(p_a, p_a^{OE})}{\partial p_a}$ satisfies the following equation:

$$\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a} J_{ia} = -\beta_i D_a\big(p_a, p_{-a}^{OE}\big)(1-\mathbb{P}_a).$$

Derivating this expression with respect to p_a we find

$$\frac{\partial^2 D_a(p_a, p_{-a}^{OE})}{\partial p_a^2} J_{ia} + \frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}\frac{\partial J_{ia}}{\partial p_a} = -\beta_i \frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}(1-\mathbb{P}_a) - \beta_i D_a\big(p_a, p_{-a}^{OE}\big)\frac{\partial(1-\mathbb{P}_a)}{\partial p_a}.$$

Using implicit differentiation on $\frac{\partial J_{ia}}{\partial p_a}$, the previous equation can be written in terms of $\frac{\partial D_a(p_a, p_{-a}^{OE})}{\partial p_a}$ and $\frac{\partial^2 D_a(p_a, p_{-a}^{OE})}{\partial p_a^2}$. Thus solving for $\frac{\partial^2 D_a(p_a, p_{-a}^{OE})}{\partial p_a^2}$ we can identify $K_{ia}(p^{OE})$ (see Lemma 2 below). A similar reasoning allows us to identify $\bar{K}_{iab}(p^{OE})$ (see Lemma 3 below).

A.2. Analysis of $\mathcal{C}_{-a}(p^{OE})$

In this section we discuss the terms $\mathcal{C}_{-a}(p^{OE})$. In particular, we show how the sign of the $\mathcal{C}_{-a}(p^{OE})$ depends on the latency functions. We consider two cases:

Case 1. $\mathcal{C}_{-a}(p^{OE}) + (1-2\mathbb{P}_a)[\frac{\partial D_a(p^{OE})}{\partial p_a}]^{-1} \geqslant 0$: Noting that $\mathcal{C}_a(p^{OE}) > 0$, it follows that

$$\Omega_{ia}\big(p^{OE}\big) + (1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1} > 0.$$

The previous condition implies that $K_{ia}(p^{OE}) > 0$, and from (19) we conclude that the profit function is concave.

Case 2. $\mathcal{C}_{-a}(p^{OE}) + (1-2\mathbb{P}_a)[\frac{\partial D_a(p^{OE})}{\partial p_a}]^{-1} < 0$: Using this condition and noting that

$$\Omega_{ia}\big(p^{OE}\big) + (1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1} = \mathcal{C}_a\big(p^{OE}\big) + \mathcal{C}_{-a}\big(p^{OE}\big) + (1-2\mathbb{P}_a)\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^{-1},$$

we get the following: If $\mathcal{C}_{-a}(p^{OE}) + (1-2\mathbb{P}_a)[\frac{\partial D_a(p^{OE})}{\partial p_a}]^{-1}$ dominates $\mathcal{C}_a(p^{OE})$, then we obtain $K_{ia}(p^{OE}) < 0$ and the concavity of the profit function will fail.[26] This implies that the existence of an OE cannot be established. Conversely, if $\mathcal{C}_{-a}(p^{OE}) + (1-2\mathbb{P}_a)[\frac{\partial D_a(p^{OE})}{\partial p_a}]^{-1}$ is dominated by $\mathcal{C}_a(p^{OE})$, it follows that $K_{ia}(p^{OE})$ is strictly positive, and by the same argument used in Case 1, we conclude the existence of an OE.

The previous analysis show us that the problem of establishing the concavity of the profit function, occurs when $\mathcal{C}_{-a}(p^{OE})$ and $(1-2\mathbb{P}_a)[\frac{\partial D_a(p^{OE})}{\partial p_a}]^{-1}$ dominate the term $\mathcal{C}_a(p^{OE})$. In other words, the complicated case is when the latency functions are such that the $K_{ia}(p^{OE})$s are strictly negative implying that the concavity of the profit functions does not hold.

A.3. Analysis of the existence and uniqueness of an OE

After this discussion we are ready to establish the technical lemmas in order to show the existence and uniqueness of an OE.

Lemma 2. *Suppose that Assumption 3 holds. Then, for all $i \neq d$, $a \in A_i^+$*

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} = -\frac{1}{D_a}\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^2\big[K_{ia}\big(p^{OE}\big) - 2\big] < 0.$$

Proof. From Lemma 1 we know that

$$\frac{\partial D_a}{\partial p_a} = \frac{-\beta_i D_a(1-\mathbb{P}_a)}{J_{ia}},$$

where $D_a \equiv D_a(p_a, p_{-a}^{OE})$ for all $a \in A_i^+$. Thus, we can rewrite the previous expression as:

$$\frac{\partial D_a}{\partial p_a} J_{ia} = -\beta_i D_a(1-\mathbb{P}_a).$$

[26] We employ the term *dominate* in an absolute value sense.

Recalling the definition of J_{ia} and taking derivative with respect to p_a we get:

$$\frac{\partial^2 D_a}{\partial p_a^2} J_{ia} + \frac{\partial D_a}{\partial p_a}\frac{\partial J_{ia}}{\partial p_a} = -\frac{1}{D_a}\frac{\partial D_a}{\partial p_a}\beta_i(1-2\mathbb{P}_a).$$

Computing the derivative $\frac{\partial J_{ia}}{\partial p_a}$, evaluating at p^{OE}, and solving for $\frac{\partial^2 D_a}{\partial p_a^2}$, we get

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} = -\frac{1}{D_a}\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\right]^2\left[K_{ia}\big(p^{OE}\big)-2\right],$$

with $0 < D_a < x_i$ and thanks to Assumption 3, $K_{ia}(p^{OE}) - 2 > 0$ for all a. Thus, we conclude that $\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} < 0$. □

Now we establish a key result to guarantee the existence of an OE. Concretely, we utilize Lemma 2 to show the quasi-concavity of $\pi_a(\cdot, p^{OE}_{-a})$.

Proposition 3. *Suppose that Assumption 3 holds. Then, for all firm a, $\pi_a(p_a, p^{OE}_{-a})$ is strictly quasi-concave in its own price p_a.*

Proof. Taking the first order condition we get that an OE satisfies:

$$\frac{\partial \pi_a(p^{OE})}{\partial p_a} = D_a\big(p^{OE}\big) + p_a^{OE}\frac{\partial D_a(p^{OE})}{\partial p_a} = 0.$$

Now, taking the second order condition evaluated at p^{OE} we find that

$$\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a^2} = 2\frac{\partial D_a(p^{OE})}{\partial p_a} + p_a^{OE}\frac{\partial^2 D_a(p^{OE})}{\partial p_a^2} < 0,$$

where the last inequality follows from Lemmas 1 and 2. Thus, we conclude that for all $a \in A$, the profit function $\pi_a(\cdot, p^{OE}_{-a})$ is strictly quasi-concave in its own price p_a.[27] □

The following proposition establishes the properties of the best response map. In particular, we establish that under Assumption 3 the best response map is convex valued.

Proposition 4. *Suppose that Assumption 3 holds, and let $(p^{OE}, D(p^{OE}))$ be a pure strategy OE. Then, at each node $i \neq d$ the best response map $B_{ia}(p^{OE}_{-a})$ is non-empty, upper semi-continuous and convex valued for all $a \in A_i^+$.*

Proof. Fix a node $i \neq d$. Using Corollary 9 we know that for every firm $a \in A_i^+$, the profit function is continuous and $S_a = [0, R_a]$ is a compact set, then there exists at least one maximizer, which implies that $B_{ia}(p^{OE}_{-a})$ is non-empty. By the Maximum Theorem, $B_{ia}(p^{OE}_{-a})$ is upper semi-continuous. The fact that $B_{ia}(p^{OE}_{-a})$ is a convex set follows from Proposition 3. □

To establish the uniqueness of an OE we use the dominant diagonal property (cf. Vives, 2001, Ch. 2). In order to apply such a property we need to establish two technical results, which are given in Lemmas 3 and 4.

Lemma 3. *For all $i \neq d$, $a \neq b \in A_i^+$:*

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b} = -\frac{1}{D_a}\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\frac{\partial D_a(p^{OE})}{\partial p_b}\right][\bar{K}_{iab} - 1] > 0.$$

Proof. From Lemma 1 we know that

$$\frac{\partial D_a}{\partial p_b} = \frac{\beta_i D_a \mathbb{P}_b}{J_{ia}},$$

where $D_a \equiv D_a(p_a, p^{OE}_{-a})$ for all $a \in A_i^+$. Thus, we can rewrite the previous expression as:

$$\frac{\partial D_a}{\partial p_b} J_{ia} = \beta_i D_a \mathbb{P}_b.$$

[27] We note that this way of proving concavity is standard. For details see Vives (2001, Ch. 2).

Recalling the definition of J_{ia} and taking derivative with respect to p_a we get:

$$\frac{\partial^2 D_a}{\partial p_a \partial p_b} J_{ia} + \frac{\partial D_a}{\partial p_a}\frac{\partial J_{ia}}{\partial p_a} = \frac{\partial D_a}{\partial p_a}\beta_i\left[\mathbb{P}_b - \frac{\mathbb{P}_a}{n_i - 1}\right].$$

Computing the derivative $\frac{\partial J_{ia}}{\partial p_a}$, evaluating at p^{OE}, and solving for $\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b}$, we get

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b} = -\frac{1}{D_a}\left[\frac{\partial D_a(p^{OE})}{\partial p_a}\frac{\partial D_a(p^{OE})}{\partial p_b}\right]\left[\bar{K}_{ib}\left(p^{OE}\right) - 1\right],$$

where $0 < D_a < x_i$ and by Assumption 3, $\bar{K}_{ib}(p^{OE}) - 1 > 0$ for all b. Thus, we conclude that

$$\frac{\partial^2 D_a(p^{OE})}{\partial p_a \partial p_b} > 0, \quad \forall a, b \in A. \qquad \square$$

Lemma 4. *For all $i \neq d$, $a, b \in A_i^+$ with $a \neq b$*

$$\sum_{b \neq a} \frac{\mathbb{P}_b}{1 - \mathbb{P}_a}\frac{\bar{K}_{iab}(p^{OE})}{K_{ia}(p^{OE})} < 1 \quad \forall p^{OE}.$$

Proof. Note that for $a \neq b$, $K_{ia}(p^{OE})$ and $\bar{K}_{iab}(p^{OE})$ can be written as:

$$K_{ia}\left(p^{OE}\right) = 1 + \frac{D_a}{J_{ia}}\bar{\Omega}_{ia} + \frac{\mathbb{P}_a}{1 - \mathbb{P}_a}, \qquad \bar{K}_{iab}\left(p^{OE}\right) = \frac{D_a}{J_{ia}}\bar{\Omega}_{ia} + \frac{\mathbb{P}_a}{\mathbb{P}_b(n_i - 1)},$$

where $\bar{\Omega}_{ia}$ is defined as:

$$\bar{\Omega}_{ia} \equiv \beta_i\left[(1 - 2\mathbb{P}_a)l'_a + \sum_{b \neq a} l'_b\left(\frac{(n_i - 1)\mathbb{P}_b - \mathbb{P}_a}{(n_i - 1)^2}\right) + D_a(1 - \mathbb{P}_a)l''_a - \sum_{b \neq a}\frac{D_a \mathbb{P}_b l''_b}{(n_i - 1)^2}\right].$$

Using this fact we get:

$$\begin{aligned}
\sum_{b \neq a} \frac{\mathbb{P}_b}{1 - \mathbb{P}_a}\frac{\bar{K}_{iab}(p^{OE})}{K_{ia}(p^{OE})} &= \sum_{b \neq a}\frac{D_a}{(1 - \mathbb{P}_a)K_{ia}(p^{OE})}\mathbb{P}_b\left(\frac{\bar{\Omega}_{ia}}{J_{ia}} + \frac{1}{D_b(n_i - 1)}\right), \\
&= \frac{D_a}{(1 - \mathbb{P}_a)K_{ia}(p^{OE})}\left(\sum_{b \neq a}\mathbb{P}_b\frac{\bar{\Omega}_{ia}}{J_{ia}} + \frac{1}{x_i}\right), \\
&= \frac{D_a}{(1 - \mathbb{P}_a)K_{ia}(p^{OE})}\left((1 - \mathbb{P}_a)\frac{\bar{\Omega}_{ia}}{J_{ia}} + \frac{1}{x_i}\right),
\end{aligned}$$

where the last equality follows because of $\sum_{b \neq a}\mathbb{P}_b = 1 - \mathbb{P}_a$. On the other hand, for $K_{ia}(p^{OE})$ we get:

$$K_{ia}\left(p^{OE}\right) = \frac{D_a}{1 - \mathbb{P}_a}\left(\frac{1}{D_a} + (1 - \mathbb{P}_a)\frac{\bar{\Omega}_{ia}}{J_{ia}}\right).$$

Combining the expressions for $K_{ia}(p^{OE})$ and $\bar{K}_{iab}(p^{OE})$, we find

$$\sum_{b \neq a} \frac{\mathbb{P}_b}{1 - \mathbb{P}_a}\frac{\bar{K}_{iab}(p^{OE})}{K_{ia}(p^{OE})} = \frac{(1 - \mathbb{P}_a)\frac{\bar{\Omega}_{ia}}{J_{ia}} + \frac{1}{x_i}}{(1 - \mathbb{P}_a)\frac{\bar{\Omega}_{ia}}{J_{ia}} + \frac{1}{D_a}}.$$

Using the fact $0 < D_a < x_i$, we conclude that $\sum_{b \neq a}\frac{\mathbb{P}_b}{1 - \mathbb{P}_a}\frac{\bar{K}_{iab}(p^{OE})}{K_{ia}(p^{OE})} < 1$. $\square$

Now we are ready to proof Theorem 3.

Proof of Theorem 3.

Existence: First, thanks to Proposition 4, the correspondence $B(p^{OE})$ is non-empty, upper semi-continuous and convex valued. Then, by Kakutani's fixed point Theorem, it follows that there exists a price vector p^{OE} such that $p^{OE} = B(p^{OE})$. Second, we show that for p^{OE} there exists an MTE given by $D(p^{OE})$, such that the condition (10) is satisfied. In particular, we show that for any node $i \neq d$ and given p^{OE}_{-a}, the firm $a \in A_i^+$ does not have a profitable deviation. In fact, noting that

$D_a(p^{OE})$ can be written as $D_a(p^{OE}) = x_i - \sum_{b \neq a} D_b(p^{OE})$ for all $b \in A_i^+$, and thanks to Proposition 1 it follows that the flow is uniquely determined, which means that firm a does not have incentive to deviate from p_a^{OE}. As this argument is valid at any node $i \neq d$, we conclude that $(p^{OE}, D(p^{OE}))$ is an OE.

UNIQUENESS: As we said before, to establish the uniqueness we apply the dominant diagonal property. Concretely, at every node $i \neq d$ and for $a, b \in A_i^+$, we compute the term:

$$-\sum_{b \neq a} \frac{\partial^2 \pi_a(p^{OE})}{\partial p_a \partial p_b} \left(\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a^2} \right)^{-1}, \quad \forall a, b \in A_i^+.$$

Using Lemmas 2 and 3 we get:

$$\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a \partial p_b} = \frac{\partial D_a(p^{OE})}{\partial p_b} \bar{K}_{iab}, \quad \text{for all } b \neq a \in A_i^+,$$
$$\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a^2} = \frac{\partial D_a(p^{OE})}{\partial p_a} K_{ia}, \quad \text{for all } a \in A_i^+.$$

Thus, we find that:

$$-\sum_{b \neq a} \frac{\partial^2 \pi_a(p^{OE})}{\partial p_a \partial p_b} \left(\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a^2} \right)^{-1} = \sum_{b \neq a} \frac{\mathbb{P}_b}{1 - \mathbb{P}_a} \frac{\bar{K}_{iab}(p^{OE})}{K_{ia}(p^{OE})}.$$

Then, thanks to Lemma 4, it follows that

$$-\sum_{b \neq a} \frac{\partial^2 \pi_a(p^{OE})}{\partial p_a \partial p_b} \left(\frac{\partial^2 \pi_a(p^{OE})}{\partial p_a^2} \right)^{-1} < 1, \quad \forall a, b \in A_i^+,$$

and we conclude that the equilibrium is unique. □

Proof of Proposition 2. Let p_{-a}^{OE} be an OE for all firms $b \neq a$. Then the best response for firm a is characterized by $\frac{\partial \pi(p_a, p_{-a}^{OE})}{\partial p_a} = 0$. Thus, it follows that p_a^{OE} being a best response to p_{-a}^{OE} must satisfy

$$D_a\big(p^{OE}\big) + p_a^{OE} \frac{\partial D_a(p^{OE})}{\partial p_a} = 0.$$

Then, using the expression for $\frac{\partial D_a(p^{OE})}{\partial p_a}$ given in Lemma 1, we find

$$p_a^{OE} = \frac{1}{\beta_i(1 - \mathbb{P}_a)} + D_a\big(p^{OE}\big)\left[l_a'\big(D_a\big(p^{OE}\big)\big) + \frac{\sum_{b \neq a} q_b l_b'(D_b(p^{OE}))}{n_i - 1} \right]. \qquad \square$$

A.4. Symmetric case

Proposition 5. *Let $(p_n^{OE}, D(p_n^{OE}))$ be a symmetric price equilibrium. Then, the following holds*

$$p_{n_i+1}^{OE} < p_{n_i}^{OE}, \quad \forall i \neq d.$$

Proof. From a symmetric pure strategy OE condition it follows that

$$p_{n_i+1}^{OE} = \frac{n_i + 1}{\beta n_i} + \frac{x_i}{n_i} l'\big(x_i/(n_i + 1)\big), \quad \forall i \neq d,$$
$$p_{n_i}^{OE} = \frac{n_i}{\beta(n_i - 1)} + \frac{x_i}{n_i - 1} l'(x_i/n_i).$$

Computing $p_{n_i+1}^{OE} - p_{n_i}^{OE}$ we get:

$$p_{n_i+1}^{OE} - p_{n_i}^{OE} = -\frac{1}{\beta n_i(n_i - 1)} + \frac{x_i}{n_i(n_i - 1)} \big[n_i\big(l'\big(x_i/(n_i + 1)\big) - l'(x_i/n_i)\big) - l'\big(x_i/(n_i + 1)\big) \big].$$

Thus, thanks to the convexity of $l(\cdot)$, the term $(l'(x_i/(n_i + 1)) - l'(x_i/n_i))$ is negative. Combining this fact with $l'(\cdot) > 0$, it follows that $p_{n_i+1}^{OE} - p_{n_i}^{OE} < 0$, or equivalently $p_{n_i+1}^{OE} < p_{n_i}^{OE}$. □

Corollary 2. *Let* $(p_n^{OE}, D(p_n^{OE}))$ *be a symmetric equilibrium. Then, the following holds*

$$\pi_a\big(D_a\big(p_{n+1}^{OE}\big)\big) < \pi_a\big(D_a\big(p_n^{OE}\big)\big), \quad \forall a \in A.$$

Proof. For all firms $a \in A$ consider the symmetric equilibriums $p_{n_i+1}^{OE}$ and $p_{n_i}^{OE}$ with the associated profits $\pi_a(D_a(p_{n+1}^{OE}))$ and $\pi_a(D_a(p_n^{OE}))$. Computing $\pi_a(D_a(p_{n+1}^{OE})) - \pi_a(D_a(p_n^{OE}))$ we get:

$$\pi_a\big(D_a\big(p_{n+1}^{OE}\big)\big) - \pi_a\big(D_a\big(p_n^{OE}\big)\big) = \frac{x_i}{n_i(n_i+1)}\big[n_i\big(p_{n_i+1}^{OE} - p_{n_i}^{OE}\big) - p_{n_i}^{OE}\big] < 0,$$

where the last inequality follows from Proposition 5. Thus we conclude that profits are decreasing in n. $\square$

Proof of Theorem 1.

<u>EXISTENCE</u>: Noting that for a symmetric OE we have that for all $i \neq d$, $a \in A$:

$$K_{ia}\big(p_n^{OE}\big) = \frac{n_i}{n_i - 1} + \frac{\beta D_a}{J_{ia}}\left(\frac{n_i - 2}{n_i(n_i - 1)}\right)\big[\big(l'(D_a)(2n_i - 1) + l''(D_a)\big)\big] > 0,$$

with $D_a = \frac{x_i}{n_i}$. Thus, we find that Assumption 3 is satisfied and the existence of a symmetric OE follows from Theorem 3.

<u>UNIQUENESS</u>: In order to show the uniqueness, note that for all $i \neq d$, $b \in A$ it holds that:

$$\bar{K}_{iab}\big(p_n^{OE}\big) = \frac{1}{n_i - 1} + \frac{\beta D_a}{J_{ia}}\left(\frac{n_i - 2}{n_i - 1}\right)\big[l'(D_a) + D_a l''(D_a)\big] > 0,$$

with $D_a = \frac{x_i}{n_i}$. In particular, we see that $K_{ia}(p^{OE}) > \bar{K}_{iab}(p_n^{OE})$, which implies that Lemma 4 applies, so we conclude that the symmetric equilibrium is unique. $\square$

Proof. Proof of Theorem 2 First, as we noted the function $\mathbb{W}(p_n^{OE})$ is strictly concave in n. Thus, taking the first order conditions and solving for n we get:

$$\frac{\partial \mathbb{W}(p_n^{OE})}{\partial n_i} = \frac{x}{\beta n_i} + \left[\frac{x_i}{n_i}\right]^2 l'(x_i/n_i) - F = 0, \quad \forall i \neq d.$$

Thus, the optimal number of firms at each node is given by

$$\forall i \neq d \quad \frac{x_i}{\beta n_i^S} + \left[\frac{x_i}{n_i^S}\right]^2 l'\big(x_i/n_i^S\big) = F \quad (S).$$

Moreover, thanks to the convexity of $l(\cdot)$ the left-hand side in (S) is a decreasing function of n_i^s, which implies that there exists a unique optimal solution n_i^S. Now considering the zero profit condition we find that $\pi_a(D_a(p_n^{OE})) = 0$ yields the following equation

$$\frac{x}{\beta(n_i^E - 1)} + \frac{x_i^2}{n_i^E(n_i^E - 1)} l'\big(x_i/n_i^E\big) = F \quad (E).$$

Once again, the convexity of the left-hand side in (E) implies that n^E is uniquely determined. Finally, from (S) and (E) it follows that $n^E > n^S$. $\square$

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